1 Vectors

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ be two vectors of $\mathbb{R}^n$. Then

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \ ,$$

If $n = 3$

$$u \times v = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = (u_2 v_3 - u_3 v_2)\mathbf{e}_x - (u_1 v_3 - u_3 v_1)\mathbf{e}_y + (u_1 v_2 - u_2 v_1)\mathbf{e}_z$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the unit vectors of the cartesian axes.

2 Topology of $\mathbb{R}^n$

1. A set is open iff it coincides with its interior.
2. A set is closed iff it contains its boundary points.
3. A set is compact iff it is closed and bounded.
4. The preimage of an open (closed) set through a continuous function is open (closed).
5. The image of a compact set through a continuous function is compact.
6. Level sets of continuous functions are always closed, as they are inverse images of singletons (which are closed sets, as they are boundary points of themselves).

3 Derivatives

Let $m, n \geq 1$ two integers, and $\Omega \subset \mathbb{R}^n$ an open set. Consider a function $f : \Omega \longrightarrow \mathbb{R}^m$.

– The existence of partial derivatives at a point $a \in \Omega$ does not even ensure that f is continuous at a (differently from the one-dimensional case).

– f is said to be Fréchet differentiable at $a \in \Omega$ iff there exists a linear function $Lf_a(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\lim_{||h||_{\mathbb{R}^n} \to 0} \frac{||f(a+h) - f(a) - Lf_a(h)||_{\mathbb{R}^m}}{||h||_{\mathbb{R}^m}} = 0 \ . \tag{3.1}$$

- For scalar functions ($m = 1$), for any $h \in \mathbb{R}^n$,

$$Lf_a(h) = \nabla f(a) \cdot h = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right) \cdot (h_1, h_2, \ldots, h_n) ,$$

  where ∇f is the gradient of f.

- For vector functions ($m > 1$), for any $h \in \mathbb{R}^n$,

$$Lf_a(h) = Df(a)h$$

  where $Df(a)$ is the Jacobian matrix, whose rows contain the gradients of the components of f.

- If $f \in C^1$ around the point $a \in \Omega$ then it is Fréchet differentiable at a. The converse is not true.