Università degli Studi di Milano

Report on the thesis
Stability in Hamiltonian Systems:
steepness and regularity in Nekhoroshev theory
by Santiago Barbieri

This thesis deals with the dynamics of close to integrable Hamiltonian systems. It contains several different results, but the major part of the thesis is devoted to a reinterpretation of the main assumption of Nekhoroshev theorem in the light of modern tools of algebraic geometry and this, besides giving a new point of view, allows to deduce new interesting results. Furthermore, similar tools also allow to make some advances on an important open conjecture in KAM theory (see below for a more precise description).

I am now going to describe more in detail the framework of the thesis and its results. First I recall that the phase space of an integrable Hamiltonian system is foliated in invariant tori, furthermore one can introduce the so called action angle variables $(I, \phi) \in \mathcal{U} \times \mathbf{T}^n$, $\mathcal{U} \subset \mathbf{R}^n$ in which the Hamiltonian becomes a function $H_0(I)$ of the actions I only. Integrable Hamiltonian systems are known to be exceptional, so it is particularly important to study the behavior of the system when one adds a perturbation. The two milestones of the theory of close to integrable Hamiltonian systems are KAM theory and Nekhoroshev's theorem. KAM theory applies to perturbations of Hamiltonian systems in which $H_0(I)$ fulfills a non-degeneracy property which is by now well understood. KAM theory ensures that the majority, in measure sense, of the invariant tori of H_0 persist when a small perturbation is added. Nekhoroshev theorem, in its original version applies to analytic perturbations of systems in which H_0 fulfills a non-degeneracy assumption called *steepness*, which is poorly understood. Nekhoroshev's theorem ensures that, up to times exponentially long with ϵ^{-1}, the actions I remain close to their initial value. ϵ is a measure of the size of the perturbation.

The main part of the thesis (Part I) deals with the problem of characterizing steepness in the light of modern algebraic geometry. I have to mention that some important results on steepness were already proved by Nekhoroshev himself (1973, 1979), Ilyashenko (1986), Niedermann (1996, 2006, 2007) and more recently by Chierchia, Faraggiana, Guzzo (2019).

The main result of Part I of the thesis is a theorem (Theorem B) giving an explicit condition ensuring steepness. This condition ensures that if certain algebraic equations involving the first r terms of the Taylor expansion of H_0 have no solution, then H_0 is steep. Theorem B also gives an estimated of the steepness indexes (important quantities related to the time of stability). This theorem also allows to deduce a few interesting corollaries. Corollary B1 is a remarkable generalization of the main result of Chierchia, Faraggiana, Guzzo; Corollary B2 gives a characterization of the semialgebraic set for which steepness is not ensured and Corollary B3 gives an interesting condition ensuring non steepness of H_0. These results are new and very interesting. It is difficult to imagine a better charachterization of steepness: I think we can finally say that thanks to these results steepness is well understood.

Part two of the thesis is devoted to the proof of a new Bernstein Ramirez inequality. This is an inequality allowing to control the sup of an analytic function on an open set through the sup over a much smaller compact set. The inequality proved in the thesis applies to functions whose graph is contained in a the zero set of a polynomial of a given degree. This kind of inequalities is useful in the study of steepness and also for the study of number of zeros of polynomials.

The third part of the thesis is devoted to the proof of a Nekhoroshev type theorem for Hölder perturbations of a steep analytic integable system. To comment on this, I recall that it is well known that the time of stability in Nekhoroshev theorem becomes a power of ϵ^{-1} for perturbations of finite smoothness. This had been proved previously for the case of quasiconvex H_0 in which a simpler proof is known to apply. In this part of the thesis Barbieri proves that in the case of perturbations of class C^ℓ the time of stability is of order $(|\ln\epsilon|^{\ell-1}\epsilon^a)^{-1}$ with $a=(\ell-1)/2n\alpha_1...\alpha_{n-2}$ and α_j the so called steepness indices of H_0. Technically the proof consists in finding an analytic approximation of the Hölder Hamiltonian and in applying the analytic Nekhoroshev theorem to the approximating system. The main new point is the technique of approximation used in the thesis, which is a refinement of the one introduced by Jackson-Moser-Zehnder. In particular Barbieri adapts it to the case of functions of action angle variables, namely functions which are periodic in the angles.

The forth part of the thesis deals with a quantitative Morse Sard inequality allowing to estimate in a quantitative way the distance between common critical levels of some functions. This result is expected to be useful in the proof of a conjecture by Arnold, Kozlov and Neishtadt on KAM theory. Such a conjecture states that the measure of the complement of the set of invariant tori should be of order ϵ if the perturbation is of order ϵ. The point is that a naive estimate would give a measure of order $\sqrt{\epsilon}$, while a mechanism found by Arnold, Kozlov and Neishtadt leads to the idea that the set should be much smaller. The conjecture has been proved by Biasco and Chierchia (2020) in a special case, but the general (and typical) case remains open. Barbieri's lemma is probably an important step for the proof of the general result.

As a conclusion I think that this is a very good thesis, which is quite impressing for the amount of results it contains and for the depth of the arguments and of the ideas presented. I surely recommend it for the defense.

4/5/2023

Dario Bambusi
Professor of Mathematical Physics
Dipartimento di Matematica "Federigo Enriques"
Via Saldini 50
20133 Milano