Report on Santiago Barbieri PhD thesis

In his thesis Santiago Barbieri studies stability properties of small perturbations of integrable Hamiltonian systems, called nearly integrable Hamiltonian systems. After the introduction of action-angle coordinates $(I_1,\ldots,I_n,\theta_1,\ldots,\theta_n)$ for the integrable part, such a Hamiltonian can be written as

$$H(\theta,I)=h(I)+\varepsilon f(\theta,I)\,, \tag{1}$$

where I is in some domain of $\mathbb{R}^n$ and θ is in the n-dimensional torus $\mathbb{T}^n=(\mathbb{R}/2\pi\mathbb{Z})^n$; $\varepsilon>0$ is the (small) size of the perturbation. The associated equations of motion are

$$\begin{cases}\dot\theta(t)=\partial_I h(I(t))+\varepsilon\partial_I f(\theta(t),I(t))\\ \dot I(t)=-\varepsilon\partial_\theta f(\theta(t),I(t))\end{cases}$$

The study of these systems has a long history and was initially motivated by problems in celestial mechanics.

For $\varepsilon=0$, only the angles move, the actions are left invariant by the Hamiltonian flow. Moreover the restriction of the flow to the invariant torus $\mathcal{T}_{I_0}:=\{I_0\}\times\mathbb{T}^n$, is linear, of vector $\nabla h(I_0)$. The question then is whether such a property (up to a change of variables) is preserved, or at least whether the actions remain confined along the flow after the perturbative part is added.

- KAM theory asserts that, if the Hamiltonian is regular enough and under a non-degeneracy hypothesis for the map $I\mapsto\nabla h(I)$, most invariant tori $\mathcal{T}_{I_0}=I_0\times\mathbb{T}^n$ of the integrable system persist (slightly deformed) as graphs over $\mathbb{T}^n$ for the perturbed one. The flow on these invariant tori remains linear. As a corollary, for $n=2$, the action variables cannot, for a simple topological reason, undergo a drift under the perturbed flow : $I(t)-I(0)=O(\sqrt{\varepsilon})$ for all time t.

- For $n\geq 3$, as it was showed by Arnold, a drift of order 1 of the action variables may occur along some orbits. Nekhoroschev proved in the $70s$ that a very long time is required for that. More precisely, by Nekhoroshev theorem, if the Hamiltonian h satisfies an appropriate property called steepness and H is real analytic, there are positive constants a,b,c such that along any orbit of the flow, $|I(t)-I(0)|\leq\varepsilon^b$ for $0\leq t\leq T(\varepsilon)=e^{c/\varepsilon^a}$.

The thesis dissertation of Santiago Barbieri deals mainly about problems related to Nekhoroshev theorem. The general introduction provides a very clear overview on the Hamiltonian systems, the integrability property, the classical perturbation theory (averaging methods), the KAM theory and the Nekhoroshev theory. Parts I and V study the steepness condition introduced by Nekhoroshev and its genericity. Part II is about Bernstein-Remez inequality for algebraic functions, a property which enters into play in the proof of the genericity of steepness, and has a number of other applications. Part III contains a stability theorem for Hamiltonians of Hölder regularity. Part IV deals with Arnold-Kozlov-Neistadt conjecture for invariant tori. The appendices contain mainly the presentation of useful tools and known results (in particular of real algebraic geometry), as well as some lemmas that are used in the different parts, with their proofs.

1. On the steepness condition

Nekhoroshev steepness condition can be stated as follow. A C^2 map $h:B^n(0,R+\delta)\subset\mathbb{R}^n\to R$ is said steep in $B(0,R)$ of steepness indices $\alpha_1,\ldots,\alpha_{n-1}\geq 1$ and steepness coefficients $C_1,\ldots,C_{n-1},\delta$ if :

1. $\inf_{I\in B(0,R)} ||\nabla h(I)|| > 0$;

2. for all $I \in B(0,R)$, for all $1 \le m \le n-1$, and for any linear subspace Γ^m of dimension m orthogonal to $\nabla h(I)$,

$$\max_{0\le\eta\le\xi} \min_{u\in\Gamma^m, ||u||=\eta} ||\Pi_{\Gamma^m}\nabla h(I+u)|| > C_m \xi^{\alpha_m} \ , \ \forall \xi \in (0,\delta] \, ,$$

where Π_{Γ^m} is the orthogonal projection on Γ_m.

This condition is a quantitative weak transversality condition designed to prevent fast drift of the action inside "resonant regions". It can be easily seen that the condition is satisfied by strictly (in the C^2-sense) convex maps h, with steepness indices $s_m = 1$, and more generally by quasi-convex maps. Outside the set of convex or quasi-convex maps, the condition is involved.

1.1 Nekhoroshev genericity theorem [Math USSR Sb., 1973] : *Let $\mathcal{P}(r,n)$ denote the set of real polynomials of n variables, of degree non greater than r. The r-jets of all the non steep functions of class C^{2r-1} in the neighborhood of a non critical point $I_0 \in \mathbb{R}^n$ are contained in a closed semi-algebraic subset $\Omega(r,n)$ of $P(r,n)$. Moreover the co-dimension of $\Omega(r,n)$ in $\mathcal{P}(r,n)$ is strictly positive for $r \ge [n^2/4]$.*

The above theorem implies both topological and measure genericity of the steepness condition. The first aim of this part of the thesis is to provide a "clarified" proof of this genericity result, exploiting new tools of real algebraic geometry. Santiago Barbieri states and proves the more specific following result.

Theorem A : *Let $s = (s_1, \ldots, s_{n-1}) \in \mathbb{N}^{n-1}$ satisfy $1 \le s_m \le r-1$ for all $m \in [\![1, n-1]\!]$ There exists a closed semi-algebraic subset $\Omega_n^{r,s}$ of $\mathcal{P}(r,n)$ such that for any $I_0 \in \mathbb{R}^n$, for any real number $\rho > 0$, for any $h \in C^{2r-1}(\overline{B}^n(I_0,\rho))$ such that $\nabla h(I_0) \neq 0$ and the r-jet $T_{I_0}(h,r,n)$ of h at I_0 is not in $\Omega_n^{r,s}$, h is steep in some open ball $B(R,I_0)$, with steepness indices $\alpha_1 = s_1$ and $\alpha_m = 2s_m - 1$ if $2 \le m \le n-1$. Moreover in $\mathcal{P}(r,n)$,*

$$codim(\Omega_n^{r,s}) \ge s_m - m(n-m-1)\, .$$

Actually Theorem A also contains uniform estimates of the steepness coefficients for the functions in some neighborhood of h. The proof is quite long and technical but clearly written and well-organized, laying the stress on the exploitation of results of real algebraic geometry, in particular Yomdin theorem [J. Complexity 2008] on the analytic reparametrization of semi-algebraic sets, which is applied, for polynomials Q of m variables satisfying $\nabla Q(0) = 0$ (which are the restrictions of $T_{I_0}(h, 2r-2, n)$ to subspaces Γ_m of dimension m in some system of coordinates) to the "thalweg"

$$\mathcal{T}(Q) = \{I \in \Gamma^m \ : \ ||I|| \le \delta \text{ and } ||\nabla Q(I)|| = \min_{u\in\Gamma^m, ||u||=||I||} ||\nabla Q(u)||\}\, .$$

1.2 There are concrete examples, e.g. in celestial mechanics, where the Hamiltonian can be approximated by an integrable one which is not convex in the action variables. Therefore it is important to have explicit criteria for steepness that can be applied in the non convex case.

Santiago Barbieri first gives an interesting characterization of the set $\Omega_n^{r,s}$ of Theorem A :

$$\Omega_n^{r,s} = \bigcup_{m=1}^{n-1} \Omega_n^{r,s_m,m}, \text{ with } \Omega_n^{r,s_m,m} = closure(\Pi_{\mathcal{P}(r,n)} Z_n^{r,s_m,m})\, ;$$

$Z_n^{r,s_m,m}$ can be seen as a subset of $\mathcal{P}(r,n) \times \mathbb{R}^K \times \mathbb{R}^n \times U_{m-1,n}$ defined by explicit algebraic equations. Here $K = (s_m - 1)(m-1)$ and $U_{m-1,n}$ is a Stiefel manifold. He proves that one can impose some bound for the component in $\mathbb{R}^n$ of the elements of $Z_n^{r,s_m,m}$, whereas there is a priori no bound for the component in $\mathbb{R}^K$.

For $m = 1$, $\Omega_n^{r,s_1,1}$ has a simple and explicit expression. For $m \geq 2$, the presence of the non bounded component makes the computation of $\Omega_n^{r,s_m,m}$ more delicate. In particular this lack of compactness does not allow to say that $\Pi_{\mathcal{P}(r,n)} Z_n^{r,s_m,m}$ is closed. Then it may be useful to distinguish for the subspaces Γ_m some properties which simplify the computations, if satisfied. First h is steep (with steepness exponent 1) on any m-dimensional subspaces Γ_m orthogonal to $\nabla h(I_0)$ and such that $D^2 h_{|\Gamma_n}(I_0)$ is non degenerate. For the m-dimensional subspaces Γ_m orthogonal to $\nabla h(I_0)$ and on which the restriction of $D^2 h(I_0)$ has a kernel of dimension 1, Santiago Barbieri shows that algebraic criteria for steepness involving the r-jet of h at I_0 can be built with algorithms using only linear operations. This is a striking observation, whose consequences are developed in this part of the dissertation.

Globally part I is a very substantial work, which gives an interesting insight into steepness property.

1.3. In Part V, Santiago Barbieri proves specific sufficient conditions of steepness, involving only the jets of order not greater than 5, for values of the dimension n between 2 and 5, It is an interesting complement to part I, illustrating in a more restricted setting the general principles stated there.

2. Bernstein-Remez inequality for algebraic functions Let $V(k,\rho)$ be the set of holomorphic functions defined on the complex disk $D(0,\rho)$ whose graph is included in an algebraic curve

$$R_S = \{(z,w) \in \mathbb{C}^2 \; : \; S(z,w) = 0\} \, ,$$

associated to a non-zero polynomial S of degree at most k, and such that over $D_\rho(0)$, R_S is the union of at most k elements that can be either vertical lines or disjoint graphs of holomorphic functions. In part II of the thesis, Santiago Barbieri proves the following result.

Theorem. *Let Ω be open in $\mathbb{C}$ and of closure included in $D(0,\rho)$. Let $K \subset \Omega$ be compact, of cardinality strictly larger than k. There exists a positive constant $C(\Omega, K)$ such that*

$$\forall f \in V(k,\rho) \, , \; \max_{\overline{\Omega}} |f| \leq C(\Omega, K) \max_K |f|$$

This is the extension of a theorem of complex analysis proved by Nekhoroshev, which was an important ingredient in his work about the genericity of steep functions. A similar result has been proved by Yomdin [Israel J. Math. 2011] by a different way. The result is far from trivial, and the proof given by Santiago Barbieri is elegant, resting on classical theorems of complex analysis.

3. Analytic smoothing and Nekhoroshev estimates for Hölder steep Hamiltonians.

Nekhoroshev theorem, originally stated for real analytic Hamiltonians, has been extended to other settings (Gevrey, C^∞, C^k), often for Hamiltonian which are convex or quasi-convex in the actions. In finite regularity, only a polynomial time of stability with respect to the inverse of the size of the perturbation can be expected. In this part, Santiago Barbieri considers Hamiltonian (1), where h is real analytic and f is of class C^l, with $l \in [n+1, \infty)$. It is assumed that h is steep, with steepness indices $\alpha_1, \ldots, \alpha_{n-1}$. The main result of this part asserts that, for ε small enough, along any orbit of the Hamiltonian flow, the action-vector remains confined in a ball of radius $C\varepsilon^b$ during a time $c\frac{1}{|\ln\varepsilon|^{l-1}}\varepsilon^a$, where

$$a = \frac{l-1}{2n\alpha_1\dots\alpha_{n-2}} + \frac{1}{2} \quad b = \frac{1}{2n\alpha_1\dots\alpha_{n-1}}\ .$$

In the convex or quasi-convex case, where $\alpha_1 = \dots = \alpha_{n-1} = 1$, the time of stability which is obtained in this result improves the known ones in finite regularity.

The use of an *ad hoc* analytic smoothing of the perturbative term h is an important ingredient of the proof. A normal form result is then applied to the regularized Hamiltonian H_s in different (resonant or non resonant) "blocks" of the phase space associated to the integrable part h. The same symplectic change of variables ψ induces a normal form for H, by the simple observation that if

$$H_s \circ \psi = h + g + f_s^*\ ,$$

then

$$H \circ \psi = h + g + [f_s^* + (H - H_s)\circ\psi]\ .$$

Here g is the resonant part and f_s^* the small remainder of the normal form for H_s. Then the second formula can be seen as a normal form for H, provided that $(H - H_s)\circ\psi$ is sufficiently small, as f_s^* is. This can be obtained with a careful choice of the parameters (in particular the parameter s of the analytic smoothing).

The method gives a flexible way to extend to less regular Hamiltonian systems stability estimates obtained in the analytic setting.

4. Quantitative Morse-Sard's Theory for nearly-integrable Hamiltonians near simple resonances.

This last part of the thesis, which describes a work in progress, is related to KAM theory. Arnold-Kozlov-Neistadt conjecture says that for a generic nearly-integrable real-analytic Hamiltonian systems $H(I,\theta) = h(I) + \varepsilon f(I,\theta)$, the complementary set of the invariant KAM tori has Lebesgue measure $O(\varepsilon)$. In fact the primary invariant tori provided by KAM theory, which are graphs over $\mathbb{T}^n$, occupy all the phase space, except a set of measure $O(\sqrt{\varepsilon})$, and one needs to consider also secondary invariant tori which are not graphs. A recent work by Biasco and Chierchia [Nonlinearity 2020] provides an essential step for the proof of the conjecture when the Hamiltonian is of the form

$$H(I,\theta) = \frac{|I|^2}{2} + \varepsilon f(\theta)\ .$$

The main aim of this work in progress is to extend this result to perturbations εf which depend on I. This part of the thesis contains fine heuristic explanations about the reasons for this conjecture and the strategy to try to prove it. A main step consist in the study of the system at simple resonances, and the generic obtention of a normal forms which allows to obtain secondary invariant KAM tori, as the ones which appear between the separatrices of a pendulum. Santiago Barbieri states in this part a result which would help to prove genericity. It roughly says that, for a domain D of $\mathbb{R}^2$ and any map $F : D \to \mathbb{R}^4$ with a prescribed bound on the derivatives up to order 5, for any $\eta > 0$, there is a subset V of $\mathbb{R}^4$ of measure not greater than η such that, for any $\lambda \in \mathbb{R}^4 \backslash V$, $||F(x) - \lambda|| \geq C\eta^{19/6}$ everywhere. Actually the result is more specific and uses a quantitative Sard Theorem due to Yomdin [Math. Ann. 1983]. It can of course be extended to other dimensions.

Conclusion. This thesis constitutes a very significant contribution to the area of nearly integrable Hamiltonian systems. It contains both new and very interesting results and new and original proofs of known results. Santiago Barbieri had to understand and implement a large scope of tools and technics in the fields of perturbation theory, real algebraic geometry, complex analysis. I am particularly impressed by the work carried out to provide a very useful insight into the steepness condition. The proofs of the results are generally detailed and very clear. Moreover each result is carefully introduced, with a presentation of the context and sometimes enlightening heuristic explanations.

I strongly recommend the defense of this thesis.

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