BERNSTEIN-REMEZ INEQUALITY FOR ALGEBRAIC FUNCTIONS: A TOPOLOGICAL APPROACH

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ABSTRACT. By taking full advantage of the structure of complex algebraic curves and by using compactness arguments, in this paper we give a self-contained proof that holomorphic algebraic functions verify a uniform Bernstein-Remez inequality. Namely, their growth over a bounded, open, complex set is uniformly controlled by their size on a compact complex subset of sufficiently high cardinality. Up to our knowledge, the first known demonstration on the existence of such an inequality for a specific subset of algebraic functions is contained in Nekhoroshev's 1973 breakthrough on the genericity of close-to-integrable Hamiltonian systems that are stable over long time. Despite its pivotal rôle, this passage of Nekhoroshev's proof has remained unnoticed so far. This work aims at extending and generalizing Nekhoroshev's arguments to a modern framework. We stress the fact that our proof is different from the one contained in Roytwarf and Yomdin's seminal work (1998), where Bernstein-type inequalities are proved for several classes of functions.

1. INTRODUCTION AND MAIN RESULT

1.1. **The Bernstein-Remez inequality.** Let $\Omega \subset \mathbb{C}$ be an open bounded domain, $\mathcal{K} \subset \Omega$ be a compact subset and let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic in Ω and continuous in its closure $\overline{\Omega}$. The *Bernstein's constant of* f with respect to $\Omega, \mathcal{K}$ is the quantity

$$\mathtt{B}(f, \mathcal{K}, \Omega) := \max_{\overline{\Omega}} |f| / \max_{\mathcal{K}} |f| \ .$$

Any family $\mathcal{F}$ of holomorphic functions defined in Ω and continuous in $\overline{\Omega}$ is said to satisfy a *uniform Bernstein-Remez inequality* if there exists $\mathtt{C}(\mathcal{K}, \Omega) > 0$ such that for all $f \in \mathcal{F}$

$$\max_{\overline{\Omega}} |f| \leq \mathtt{C}(\mathcal{K}, \Omega) \max_{\mathcal{K}} |f| \qquad \text{or, equivalently, if} \qquad \sup_{f \in \mathcal{F}} \mathtt{B}(f, \mathcal{K}, \Omega) \leq \mathtt{C}(\mathcal{K}, \Omega) \ .$$

The term *Bernstein-Remez inequality* is used in order to avoid confusion with other sorts of Bernstein's inequalities that involve derivatives or primitives (see e.g. [19]).

The Bernstein-Remez inequality and the existence of families verifying a uniform estimate of this kind turn out to be important in many areas of mathematics. Without pretending to make a complete survey on the subject, we observe that these kind of estimates appear in the study of the local behavior of certain holomorphic functions (see e.g. [39], [9], [31], [16], [11], [36], [17]), in questions related to the second part of Hilbert's 16th problem (see e.g. [22], [9], [27], [10], [21]), in the study of special classes of ODEs (see e.g. [28]) and subelliptic PDEs (see e.g. [19], [20]), as well as in potential theory (see e.g. [38], [12]) and in dynamical systems when investigating questions related to entropy (see e.g. [40]).

In this article, we are interested in finding a family of functions verifying a uniform Bernstein-Remez inequality. Namely, by extending a strategy due to Nekhoroshev [32] and that is different from the known demonstrations in this field (see [37], [8], [42], [12]), with the above notations we shall prove the following. If

(1) the graph of f solves the algebraic equation $S(z, f(z)) = 0$ for some non-zero polynomial $S \in \mathbb{C}[X, Y]$ of degree k;
(2) the algebraic curve of S over Ω is given by the union of vertical lines of the form $\{(z, w) \in \mathbb{C}^2 \mid z = z_*\}$ together with disjoint graphs of holomorphic functions over Ω;
(3) the cardinality of $\mathcal{K}$ is strictly greater than k;

then the Bernstein's constant of f w.r.t. $\Omega, \mathcal{K}$ depends on k but is independent of f.

Before stating this result more rigorously (see Theorem 1.2), let us discuss our motivation for developing this subject.

1.2. **Rôle in Hamiltonian dynamics and Nekhoroshev theory.** The authors discovered the Bernstein-Remez inequality during the investigation of an important result of Hamiltonian dynamics. However, before describing the key rôle played by the Bernstein-Remez estimate in this field, we make a short review of some general results which are helpful in order to make the context clear to the reader.

Namely, Hamiltonian formalism is the natural setting appearing in the study of many physical systems. In the simplest case, we consider the motion of a point on a Riemannian manifold $\mathcal{M}$, called configuration manifold, governed by Newton's second law ($\ddot{q} = -\nabla U(q)$ for a potential function U in the euclidean case, with q a system of local coordinates for $\mathcal{M}$). This system can be transformed by duality thanks to Legendre's transformation and reads

$$\dot{p} = -\partial_q H(p, q) \quad , \qquad \dot{q} = \partial_p H(p, q) \quad ,$$

where $H(p, q)$ is a real differentiable function on the cotangent bundle $T^*\mathcal{M}$, classically called Hamiltonian, and p is the coordinate conjugated to q. Systems integrable by quadrature are an important class of Hamiltonian systems. A Hamiltonian system depending on $2n$ variables (n degrees of freedom) is said to be integrable in the sense of Arnol'd-Liouville if it can be conjugated to a Hamiltonian system on the cotangent bundle of the n-dimensional torus $\mathbb{T}^n$, whose equations of motion take the form

$$\dot{I} = -\partial_\vartheta h(I) = 0 \quad , \qquad \dot{\vartheta} = \partial_I h(I) \quad ,$$

where $(I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ are called action-angle coordinates. Therefore, the phase space for an integrable system is foliated by invariant tori carrying the linear motions of the angular variables (called quasi-periodic motions). Integrable systems are exceptional, but many important physical problems are governed by Hamiltonian systems which are close to integrable. Namely, the dynamics of a nearly-integrable Hamiltonian system is described by a Hamiltonian function whose form in action-angle coordinates $(I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ reads

$$H(I, \vartheta) := h(I) + \varepsilon f(I, \vartheta) ,$$

where ε is a small parameter. The structure of the phase space for this kind of systems can be inferred with the help of Kolmogorov-Arnol'd-Moser (KAM) theory. Namely, under a generic non-degeneracy condition for h, a Cantor set of large measure of invariant tori carrying quasi-periodic motions for the integrable flow persists under a suitably small perturbation (see e.g. ref. [2], [14]).

For systems with three or more degrees of freedom, KAM theory yields little information about trajectories lying in the complementary of such Cantor set, where instabilities may occur (see e.g. ref. [1]). However, in a series of articles published during the seventies (see ref. [33], [34], or [25], [4] for a more modern presentation), Nekhoroshev proved an effective result of stability for all initial conditions holding over a time which is exponentially long in the inverse of the size ε of the perturbation, provided that the Hamiltonian is analytic and that its integrable part satisfies a generic transversality property known as *steepness*.

In order to introduce the steepness property, we fix a positive integer $n \geq 2$ and we indicate by $B^n(0,R)$ the real n-dimensional ball of radius R centered at the origin. Then, we have

Definition 1.1 (Steepness)**.** Fix $\delta > 0$, $R > 0$. A C^2 function $h : B^n(0, R+2\delta) \to \mathbb{R}$ is steep in $B^n(0,R)$ with steepness indices $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1} \geq 1$ and *steepness coefficients* $C_1, \dots, C_{n-1}, \delta$ if:

(1) $\inf_{I \in B^n(0,R)} ||\nabla h(I)|| > 0$;

(2) for any $I \in B^n(0,R)$, for any integer $1 \leq m < n$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla h(I)$ and endowed with the induced euclidean metric, one has:

$$\max_{0 \leq \eta \leq \xi} \min_{u \in \Gamma^m,\, ||u||_2 = \eta} || \pi_{\Gamma^m} \nabla h(I+u) || > C_m \xi^{\alpha_m}, \quad \forall \xi \in (0,\delta], \tag{1.1}$$

where π_{Γ^m} stands for the orthogonal projection on Γ^m.

Remark 1.1. Since in definition 1.1 the subspace $\Gamma^m \subset \mathbb{R}^n$ is endowed with the induced metric, for all $u \in \Gamma^m$ one has $||\pi_{\Gamma^m} \nabla h(I+u)|| = ||\nabla(h|_{I+\Gamma^m})(I+u)||$, where $h|_{I+\Gamma^m}$ indicates the restriction of h to the affine subspace $I + \Gamma^m$.

Remark 1.2. It is worth mentioning that a real-analytic function is steep if and only if it has no isolated critical points and if any of its restrictions to any affine proper subspace has only isolated critical points (see [26] and [35]).

With this notion, Nekhoroshev's effective result of stability reads

Theorem 1.1 (Nekhoroshev, 1977)**.** *Consider a nearly-integrable system with Hamiltonian* $H(I,\vartheta) := h(I) + \varepsilon f(I,\vartheta)$ *analytic in some complex neighborhood of* $B^n(0,R) \times \mathbb{T}^n$*, and assume that* h *is steep. Then, there exist positive constants* $a, b, \varepsilon_0, C_1, C_2, C_3$ *such that, for any* $\varepsilon \in [0, \varepsilon_0)$ *and for any initial condition not too close to the boundary, one has* $|I(t) - I(0)| \leq C_2 \varepsilon^a$ *for any time* t *satisfying* $|t| \leq C_1 \exp\left(C_3/\varepsilon^b\right)$.

Nekhoroshev also proved in [32] that the steepness condition is generic, both in measure and in topological sense: for a sufficiently high positive integer r, the Taylor polynomials of order less or equal than r of non-steep functions are contained in a semi-algebraic[1] set having positive codimension in the space of polynomials of order bounded by r. Hence,

[1] A subset of $\mathbb{R}^n$ is said to be semi-algebraic if it can be determined by a finite number of polynomial equalities and inequalities.

steep functions are characterised by the fact that their Taylor polynomials satisfy suitable algebraic conditions (see [34] and [3]). Although these results have been studied and extended for more than forty years (so that *Nekhoroshev Theory* is a classic subject of study in the dynamical systems community), the proof of the genericity of steepness has remained, up to now, largely unstudied and poorly understood. This is certainly due to the fact that such a demonstration does not involve any arguments of dynamical systems, but combines quantitative reasonings of real-algebraic geometry and complex analysis. It is precisely in those reasonings that the Bernstein-Remez inequality plays a major rôle.

1.2.1. *The rôle of Bernstein-Remez inequality.* A crucial step in Nekhoroshev's proof of the genericity of steepness consists in considering, for any fixed polynomial $P \in \mathbb{R}[X_1, \dots, X_m]$, the semi-algebraic set - called thalweg nowadays (see [6]) - defined by

$$\mathcal{T}_P \subset \mathbb{R}^m := \{u \in \mathbb{R}^m | \ ||\nabla P(u)|| \le ||\nabla P(v)|| \ \forall v \in \mathbb{R}^m \text{ s.t. } ||u|| = ||v||\} \ . \tag{1.2}$$

Remark 1.3. In order to grasp why this kind of set is interesting in the study of the genericity of steepness, it is worth comparing (1.2) with (1.1) from a heuristic point of view. Infact, in Definition 1.1, one is interested in controlling quantitatively the projection of the gradient of the function h on any affine subspace Γ^m which is orthogonal to $\nabla h(I)$. Fixing Γ^m and taking Remark 1.1 into account, if one approximates the restriction $h|_{I+\Gamma^m}$ by its Taylor polynomial $P_{h,I+\Gamma^m}$ at a suitable order, then studying the locus

$$\left\{ I + u \in I + \Gamma^m \text{ s.t. } ||\nabla P_{h,I+\Gamma^m}(I+u)|| = \min_{w \in \Gamma^m, ||w|| = \eta} ||\nabla P_{h,I+\Gamma^m}(I+w)|| \right\}$$

amounts to studying the set $\mathcal{T}_{P_{h,I+\Gamma^m}}$ in (1.2), where we have identified $P \equiv P_{h,I+\Gamma^m}$.

Nekhoroshev shows that, for any open ball $\mathcal{B} \subset \mathbb{R}^m$ and for any given polynomial P, the intersection $\mathcal{T}_P \cap \mathcal{B}$ contains a real analytic curve C such that both the distance between the extremities of C and the complex analyticity width of its parametrization admit a lower bound that depends only on m and on the degree of the polynomial P. More specifically, C can be parametrized by algebraic functions. The existence of a uniform Bernstein-Remez inequality (also proved in [32] in a less general context than the one we consider in the following paragraphs) ensures uniform upper bounds on the derivatives of these charts.

The uniform control on the parametrization of the curve C is unavoidable in [32], since it ensures that - for a smooth function - steepness is an open property which can be determined by the Taylor expansion at a certain order (we have a "finite-jet" determinacy of steepness). Namely, with the setting of Definition 1.1, if for any m-dimensional subspace Γ^m orthogonal to $\nabla h(I)$ the Taylor polynomial $P_{h,I+\Gamma^m}$ verifies condition (1.1), then the uniform control on the derivatives of the curve C contained in the thalweg $\mathcal{T}_{P_{h,I+\Gamma^m}}$ ensures that estimate (1.1) is verified uniformly also by polynomials belonging to a neighborhood of $P_{h,I+\Gamma^m}$.

In this way, the study of the genericity of steepness is reduced to the study of uniform lower estimates of the kind (1.1) in a finite-dimensional setting which involves polynomials of bounded order. This aspect, together with additional technicalities which will not be discussed here, is crucial in order to prove that the Taylor polynomial of suitably high order of non-steep functions are contained in a semi-algebraic set having positive codimension in the space of polynomials of bounded order. This aspect will be investigated and specified in a forthcoming paper of the first author.

1.3. **Rôle in semi-algebraic geometry.** Actually, the result about the thalweg described above is a particular case of a general theorem about analytic reparametrizations of semi-algebraic sets. Namely, in refs. [41] and [43], Yomdin has shown that - with the exception of a small part - any two-dimensional semi-algebraic set can be covered by the images of a finite number of real-analytic, algebraic charts of the interval $[-1, 1]$. Moreover, thanks to the existence of a Bernstein-Remez inequality for algebraic functions, one has a bound over the size of all the derivatives of these charts that depends only on the order of the derivation and on the degrees of the polynomials involved in the definition of the considered semi-algebraic set. This is a partial extension of the theorem (called Algebraic Lemma) about the C^k-reparametrization of semi-algebraic sets proved independently by Yomdin and Gromov (see [40], [24], [13]). The analytic reparametrization in [41] result has recently been generalized (see [5] and [15]) to higher dimensional sets with more general structures than semi-algebraic, which allows for important applications in arithmetics.

From a more general point of view, the steepness condition is introduced to prevent the abundance of rational vectors on certain sets. In particular, deep applications of the controlled analytic parametrizations of semi-algebraic sets - yielding bounds on the number of integer points in semi-algebraic sets - are given in [5] and [15]. Along these lines of ideas, the Yomdin-Gromov algebraic lemma with tame parametrizations of semi-algebraic sets (see [40], [24]) was used by Bourgain, Goldstein, and Schlag [7] to bound the number of integer points in a two-dimensional semi-algebraic set.

1.4. **Different strategies of proof.** In ref. [32], Nekhoroshev proves the existence of a Bernstein-Remez inequality for algebraic functions in his specific problem, by exploiting the properties of complex algebraic curves and by making an intensive use of complex analysis (especially, of compactness arguments exploiting Montel's Theorem). The original statements are difficult to disentangle from the context of the genericity of steepness and their proofs are very sketchy. The existence of Bernstein-Remez inequalities in more general situations has been proved in relatively more recent times by Roytwarf-Yomdin [37], Briskin-Yomdin [8], and Yomdin [42], by combining the controlled growth of the Taylor coefficients of p-valent functions[2] together with arguments of analytic geometry. Moreover, in a closely related problem, Brudnyi has proved in [12] the existence of Bernstein-Remez inequalities for polynomials restricted to graphs of multivariate holomorphic functions.

Nekhoroshev's different strategy of proof is briefly mentioned in [37] (p. 848), without quoting [32]. The strategy of Brudnyi's work [12] relies mainly on potential theory. In particular, Lemma 2.1 in [12] contains a reasoning similar to a minor part of Nekhoroshev's reasonings in combination with a result by Sadullaev (see [38]). However, the overall framework of [12] is very different from Nekhoroshev's one, and the core of Nekhoroshev's arguments does not appear (in particular, Lemma 4.2 below). In conclusion, so far we have

[2]An analytic function over a disc is said to be p-valent if either it is constant or each element of $\mathrm{Im}(f)$ is the image of at most k points. Any algebraic function f satisfying $S(z, f(z)) = 0$ for some non-zero polynomial $S \in \mathbb{C}[X, Y]$ of degree k is k-valent (Lemma 3.1).

not been able to find any reference that shows Nekhoroshev's proof in detail except for the original paper (see [32], Lemma 5.1, p.446).

This is our motivation for a short, self-contained exposition of Nekhoroshev's proof relying on arguments complex analysis. Actually, Nekhoroshev [32] shows the existence of a Bernstein-Remez inequality only in the case in which the compact set $\mathcal{K}$ is a real segment and the considered algebraic functions have a particular form, since this is sufficient for his purposes. Here, we extend this strategy by considering any compact set $\mathcal{K}$ of high enough cardinality and we get rid of the additional conditions on the form of the algebraic functions.

Nekhoroshev's approach presents two drawbacks. It does not allow for quantitative estimates for the Bernstein constants as in [37] and [42]. Moreover, we were not able to prove a Bernstein-Remez inequality for an algebraic function on its maximal disk of regularity, what is obtained in [37] and is called structural inequality, but only for the maximal disk of regularity of all the algebraic functions associated to the considered polynomial. However, these two points are not mandatory for applications of the Bernstein-Remez inequality to Nekhoroshev's arguments on the thalweg and, more generally, to describe the overall structure of semi-algeraic sets (see [41]).

Finally, as it was already known in [32] and is central in [37], the existence of uniform Bernstein's constants implies uniform bounds on the Taylor coefficients of algebraic functions. In this spirit, we shall also state a result of this kind in Corollary 2.1.

1.5. **Main result.** By the discussion above, it is of crucial importance to find classes of functions admitting a uniform bound on their Bernstein's constants, and thus satisfying a uniform Bernstein-Remez inequality. In this paper we will establish the existence of a uniform Bernstein-Remez inequality for the following class of analytic-algebraic functions:

Definition 1.2. Consider $k \in \mathbb{N}$, $\rho > 0$ and denote by $\mathcal{D}_\rho(0)$ the open complex disk of radius ρ centered at the origin.

We indicate by $\mathcal{V}(k,\rho)$ the set of functions f that satisfy:

(1) f is holomorphic over $\mathcal{D}_\rho(0)$;
(2) The graph of f is included in an algebraic curve
$$\mathtt{R}_S := \{(z,w) \in \mathbb{C}^2 \,:\, S(z,w) = 0\}$$
associated to a non-zero polynomial $S \in \mathbb{C}[z,w]$ of degree at most k, hence
$$S(z, f(z)) = 0 \qquad \text{for } z \in \mathcal{D}_\rho(0) \;;$$
(3) The algebraic curve $\mathtt{R}_S$ is such that $\mathtt{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$ is the union of at most k elements that can be either vertical lines of the form $\{(z,w) \in \mathbb{C}^2 \mid z = z_*\}$ or disjoint graphs of holomorphic functions over $\mathcal{D}_\rho(0)$.

The functions in the class $\mathcal{V}(k,\rho)$ verify the following

Theorem 1.2 (Main result)**.** *With the notations of Definition 1.2, consider a compact set $\mathcal{K} \subset \mathcal{D}_\rho(0)$ satisfying:*

$$0 \in \mathcal{K} \text{ and } \operatorname{card}(\mathcal{K}) > k. \tag{1.3}$$

Then, the functions of the family $\mathcal{V}(k,\rho)$ verify a uniform Bernstein-Remez inequality with respect to $\mathcal{K}$ and to any open set Ω such that $\mathcal{K} \subset \Omega$ and $\overline{\Omega} \subset \mathcal{D}_\rho(0)$.

Consequently, there exists a number $\mathtt{C} = \mathtt{C}(k, \rho, \mathcal{K}, \Omega) > 0$ *such that, for any* $f \in \mathcal{V}(k, \rho)$*, one has:*

$$\max_{z \in \overline{\Omega}} |f(z)| \leq \mathtt{C} \max_{z \in \mathcal{K}} |f(z)| \,.$$

This theorem has been demonstrated by Briskin-Yomdin and Roytwarf-Yomdin in refs. [8]- [37] in the cases where $\mathcal{K} = [-\rho', \rho'] \subset \mathbb{R}$ or $\mathcal{K} = \overline{\mathcal{D}}_{\rho'}(0) \subset \mathbb{C}$, and $\Omega = \mathcal{D}_{\rho''}(0) \subset \mathbb{C}$, with $0 < \rho' < \rho'' < \rho$. Moreover, the authors obtain quantitative estimates on the upper bound $\mathtt{C}(k, \rho', \rho'', \mathcal{K})$ for the Bernstein's constant and they generalize these results to relevant cases of algebraic families of holomorphic functions. More recently, these estimates have been extended by Yomdin and Friedland to the case of a discrete compact $\mathcal{K}$ of sufficiently high cardinality in refs. [42] and [23], thanks to the introduction of a geometric invariant related to entropy.

This paper is organized as follows: section 2 contains the mathematical setting and the results, whereas section 3 contains their proofs. Section 4 is devoted to the proof of some technical lemmas that are used in section 3 and is the "core" of Nekhoroshev's strategy (especially Lemma 4.2). Finally, we have relegated to the appendices the statements of some auxiliary results that are used throughout the paper.

2. Setting and other results

2.1. **Setting.** For any $r > 0$ and any $z_0 \in \mathbb{C}$, we denote by $\mathcal{D}_r(z_0)$ the open complex disk centered at z_0 and by $\overline{\mathcal{D}}_r(z_0)$ its closure.

$\mathbb{C}[z, w]$ indicates the ring of polynomials of two variables over the complex field. Throughout this paper, we will often identify $\mathbb{C}[z, w]$ with $\mathbb{C}[z][w]$, the ring of complex polynomials in w over the ring of polynomials of the complex variable z.

For $k \in \mathbb{N}$, we indicate by $\mathcal{Q}(k) \subset \mathbb{C}[w]$ and $\mathcal{P}(k) \subset \mathbb{C}[z, w]$ respectively the subspaces of complex polynomials in one and two variables having degree inferior or equal to k. Since $\mathcal{Q}(k), \mathcal{P}(k)$ are finite-dimensional, they can be equipped with an arbitrary norm.

2.2. **Other results.** With the notations of Theorem 1.2, we consider the following class of functions:

Definition 2.1. For $k \in \mathbb{N}$ and $\rho > 0$, we denote by $\mathcal{V}_0(k, \rho) \subset \mathcal{V}(k, \rho)$ the subset of those functions $g \in \mathcal{V}(k, \rho)$ that satisfy $g(0) = 0$.

The functions of the family $\mathcal{V}_0(k, \rho)$ belong to the same Bernstein's class w.r.t. the sets Ω and $\mathcal{K}$ of Theorem 1.2. Namely, one has:

Theorem 2.1. *Consider an open set* Ω *satisfying* $\overline{\Omega} \subset \mathcal{D}_\rho(0)$ *and* $\mathcal{K} \subset \Omega$ *a compact set satisfying card* $\mathcal{K} > k$*. There exists a number* $\mathtt{C}_0 = \mathtt{C}(k, \rho, \mathcal{K}, \Omega) > 0$ *that bounds uniformly the Bernstein's constants of the elements of* $\mathcal{V}_0(k, \rho)$*, i.e.:*

$$\text{for any } g \in \mathcal{V}_0(k, \rho), \text{ one has } \max_{z \in \overline{\Omega}} |g(z)| \leq \mathtt{C}_0 \max_{z \in \mathcal{K}} |g(z)|.$$

Remark 2.1. The hypothesis $0 \in \mathcal{K}$ of Theorem 1.2 is unnecessary in Theorem 2.1.

Theorem 1.2 is a consequence of Theorem 2.1 since, for any $f \in \mathcal{V}(k,\rho)$, the function $g(z) := f(z) - f(0)$ belongs to the class $\mathcal{V}_0(k,\rho)$ and Theorem 2.1 ensures:

$$\begin{aligned}\max_{\overline{\Omega}} |f| \leq & |f(0)| + \max_{\overline{\Omega}} |g| \leq |f(0)| + \mathtt{C}_0 \max_{\mathcal{K}} |g| \\ \leq & |f(0)| + \mathtt{C}_0 |f(0)| + \mathtt{C}_0 \max_{\mathcal{K}} |f| = (1+2\mathtt{C}_0) \max_{\mathcal{K}} |f| := \mathtt{C} \max_{\mathcal{K}} |f|\end{aligned}$$

where the last estimate comes from the hypothesis $0 \in \mathcal{K}$.

This concludes the proof of Theorem 1.2

Theorem 2.1 is also the cornerstone which allows one to prove a uniform upper bound on the Taylor coefficients of functions in $\mathcal{V}_0(k,\rho)$. More specifically, we introduce the following class of bounded algebraic functions:

Definition 2.2. With the previous notations, for any $M \geq 0$ and for any compact $\mathcal{K} \subset \mathcal{D}_\rho(0)$, we denote by $\mathcal{U}(k,\rho,\mathcal{K},M)$ the subset of those functions $g \in \mathcal{V}_0(k,\rho)$ that verify $\max_{\mathcal{K}} |g| = M$.

Hence, we have $\mathcal{V}_0(k,\rho) = \cup_{M\geq 0} \mathcal{U}(k,\rho,\mathcal{K},M)$.

The functions in $\mathcal{U}(k,\rho,\mathcal{K},M)$ satisfy a generalized uniform Cauchy inequality, namely

Corollary 2.1. *Under the additional assumption card $\mathcal{K} > k$, there exists a constant $K = K(k,\rho,\mathcal{K})$ such that, for any function $g \in \mathcal{U}(k,\rho,\mathcal{K},M)$, the coefficients of the Taylor series*

$$g(z) = \sum_{j=1}^{+\infty} a_j z^j \quad (with\ g(0) = 0) \tag{2.1}$$

satisfy the uniform inequality

(1)

$$|a_j| \leq K(k,\rho,\mathcal{K})M \qquad \textit{if } \rho > 1\ ;$$

(2) for any number $m > 1$

$$|a_j| \leq K(k,m,\mathcal{K})M \left(\frac{m}{\rho}\right)^j \qquad \textit{if } \rho \leq 1\ .$$

Remark 2.2. This result is stated and used in [32] in the particular case where $\rho > 1$, $\mathcal{K} = [0,\lambda] \subset \mathbb{R}$, $M(\lambda) = \lambda$ and $\lambda > 0$. The equivalence between a uniform bound on the growth of the Taylor coefficients and the Bernstein-Remez inequality is central in [37].

Theorem 2.1 and Corollary 2.1 will be proved in the next section.

3. Proof of the main results

We first need the following standard lemma:

Lemma 3.1. *With the notations of the previous section, an analytic-algebraic function f, associated to a polynomial $S \in \mathbb{C}[z,w]$ of degree $k \in \mathbb{N}$, is k-valent: that is, if f is not constant then each element of* $\operatorname{Im}(f)$ *is the image of at most k points. Consequently, if f is not identically zero, then f cannot be identically zero over any set $\mathcal{K}$ included in the domain of definition of f such that* $\operatorname{Card}(\mathcal{K}) > k$.

Proof. Assume, by contradiction, that f is non-constant and that there exists $w_0 \in \mathrm{Im}(f)$ which is the image of at least $p > k$ points. The polynomial $S^{w_0}(z) := S(z, w_0)$ would admit $p > k$ roots while $\deg(S^{w_0}) \le k$ by hypothesis. The Fundamental Theorem of Algebra ensures that S^{w_0} must be identically zero and one has the factorization $S(z,w) = (w-w_0)^\alpha \hat{S}(z,w)$, where $\alpha \in \{1, ..., k\}$, while $\hat{S}$ cannot be divided by $(w-w_0)$ in $\mathbb{C}[z,w]$. Since f is analytic and not constant, the preimage $f^{-1}(\{w_0\})$ is a discrete set and the graph of f must satisfy $\hat{S}(z, f(z)) = 0$ outside of $f^{-1}(\{w_0\})$. By continuity, one has $\hat{S}(z, f(z)) = 0$ on the whole domain of definition of f since $f^{-1}(\{w_0\})$ is discrete. But $\deg \hat{S}^{w_0} \le k$, with $\hat{S}^{w_0}(z) := \hat{S}(z, w_0)$, and $\hat{S}^{w_0}$ admits more than k roots, hence the previous argument ensures that $\hat{S}$ can be divided by $(w - w_0)$, in contradiction to construction.

Moreover, if $f \not\equiv 0$, then 0 admits at most k inverse images by f, and f cannot be identically null over any set $\mathcal{K}$ included in the domain of definition of f and satisfying $\operatorname{card} \mathcal{K} > k$. □

Consequently - without any loss of generality - in Theorem 2.1 we can assume $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ according to Definition 2.2 (hence $g \in \mathcal{V}_0(k, \rho)$ and $\max_{\mathcal{K}} |g| = 1$) since, if this is not the case, it suffices to consider $g/\max_{\mathcal{K}} |g|$.

Then, we define the following set:

Definition 3.1. $\mathcal{A} := \mathcal{A}(\mathcal{K}, k, \rho)$ denotes the set of those polynomials $S \in \mathcal{P}(k)\backslash\{0\}$ whose algebraic curve $\mathtt{R}_S := \{(z,w) \in \mathbb{C}^2 : S(z,w) = 0\}$ satisfies

(1) $\mathtt{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$ is the union of at most k elements that can be either vertical lines of the form $\{(z,w) \in \mathbb{C}^2 \mid z = z_*\}$ or disjoint graphs of holomorphic functions over $\mathcal{D}_\rho(0)$;

(2) there exists $g_S \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ whose graph is contained in $\mathtt{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$.

Remark 3.1. For any $S \in \mathcal{A}$, the function g_S is unique since the graphs contained in the algebraic curve of S are disjoint over $\mathcal{D}_\rho(0)$ and the value $g_S(0) = 0$ is fixed.

The central property in the proof of Theorem 2.1 is the following

Lemma 3.2. *$\mathcal{A} \cup \{0\}$ is closed in $\mathcal{P}(k)$ and, for any open set Ω satisfying $\mathcal{K} \subset \Omega$, $\overline{\Omega} \subset \mathcal{D}_\rho(0)$, the function*

$$\mu_\Omega : \mathcal{A} \longrightarrow \mathbb{R} \qquad S \longmapsto \max_{\overline{\Omega}} |g_S|$$

is continuous.

We shall relegate the proof of Lemma 3.2 to the next section and we shall exploit its statement here to prove Theorem 2.1 and Corollary 2.1.

Proof. (Theorem 2.1)

By Definitions 2.1, 2.2 and 3.1, we can associate to any $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ a polynomial $S \in \mathcal{A}$ such that $g = g_S$. A standard combinatorial computation yields that $\mathcal{P}(k)$ is isomorphic to $\mathbb{C}^m$, with $m = (k+1)(k+2)/2$. It is also easy to see that for any polynomial $S \in \mathcal{A}$ and for any $c \in \mathbb{C}\backslash\{0\}$ the polynomial $S' = cS$ belongs to $\mathcal{A}$ and $g_{S'} \equiv g_S$, so that it makes sense to pass to the projective space

$$\mathbb{CP}^{m-1} := \{\mathbb{C}^m\backslash\{0\}\}/\{\mathbb{C}\backslash\{0\}\} \quad , \qquad \pi : \mathbb{C}^m\backslash\{0\} \longrightarrow \mathbb{CP}^{m-1} ,$$

where π denotes the standard canonical projection inducing the quotient topology in $\mathbb{CP}^{m-1}$. Moreover, for any open set Ω satisfying $\mathcal{K} \subset \Omega$, $\overline{\Omega} \subset \mathcal{D}_\rho(0)$, the function

$$\hat{\mu}_\Omega : \pi(\mathcal{A}) \longrightarrow \mathbb{R} \quad , \qquad \pi(S) \longmapsto \max_{\overline{\Omega}} |g_S|$$

is well defined and continuous by Lemma 3.2. To prove the latter claim, take a closed set $\mathcal{E} \subset \mathbb{R}$ and consider its inverse image $\hat{\mu}_\Omega^{-1}(\mathcal{E}) = \pi(\mu_\Omega^{-1}(\mathcal{E}))$. Since μ_Ω is continuous, $\mu_\Omega^{-1}(\mathcal{E})$ is closed in $\mathcal{A}$ for the induced topology. By Lemma 3.2, $\mathcal{A} \cup \{0\}$ is closed in $\mathbb{C}^m$, so that $\mathcal{A}$ is closed in $\mathbb{C}^m \backslash \{0\}$. Hence, $\mu_\Omega^{-1}(\mathcal{E})$ is closed in $\mathbb{C}^m \backslash \{0\}$. Since μ_Ω is invariant if its argument is multiplied by a complex non-zero constant, $\mu_\Omega^{-1}(\mathcal{E})$ is saturated and one has $\pi^{-1}(\pi(\mu_\Omega^{-1}(\mathcal{E}))) = \mu_\Omega^{-1}(\mathcal{E})$. Consequently, the set $\pi(\mu_\Omega^{-1}(\mathcal{E})) = \hat{\mu}_\Omega^{-1}(\mathcal{E})$ is closed for the quotient topology because its inverse image w.r.t. π is closed. This proves the continuity of $\hat{\mu}_\Omega$.

Moreover, since $\mathcal{A}$ is closed and saturated in $\mathbb{C}^m \backslash \{0\}$, $\pi(\mathcal{A})$ is closed in $\mathbb{CP}^{m-1}$ and the compactness of $\mathbb{CP}^{m-1}$ ensures that $\pi(\mathcal{A})$ is compact. By continuity of $\hat{\mu}_\Omega$, the image $\hat{\mu}_\Omega(\pi(\mathcal{A}))$ is a compact subset of $\mathbb{R}$, hence bounded. Therefore, there exists a constant $\mathtt{C}(k, \rho, \mathcal{K}, \Omega)$ such that for any $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ one has

$$\max_{\overline{\Omega}} |g| = \frac{\max_{\overline{\Omega}} |g|}{\max_{\mathcal{K}} |g|} \leq \mathtt{C}(k, \rho, \mathcal{K}, \Omega)$$

and this concludes the proof. □

Proof. (Corollary 2.1)

Since g is non identically zero over $\mathcal{K}$ (see Lemma 3.1), we can consider the function g/M and we are reduced to the case $M = 1$.

For $\rho > 1$, the statement is a consequence of the Cauchy's estimate and of Theorem 2.1 applied to $\Omega = \mathcal{D}_1(0)$ and $\mathcal{K}$.

In case $\rho \leq 1$, for any fixed $m > 1$ one considers the function

$$g_m(z) := g\left(\frac{\rho}{m} z\right) := \sum_{j=1}^{+\infty} c_j z^j = \sum_{j=1}^{+\infty} a_j \left(\frac{\rho}{m} z\right)^j$$

analytic in $\mathcal{D}_m(0)$ and belonging to $\mathcal{U}(k, m, \mathcal{K}_m, 1)$, where

$$\mathcal{K}_m := \{z \in \mathcal{D}_m(0) : \frac{\rho}{m} z \in \mathcal{K}\}$$

satisfies $\operatorname{card} \mathcal{K}_m > k$ since $\mathcal{K}$ does.

Since the convergence radius of g_m is $m > 1$, the statement holds for this function and there exists a constant $K(k, m, \mathcal{K})$ such that

$$|c_j| \leq K(k, m, \mathcal{K}) \qquad \forall\, j \in \mathbb{N},$$

which implies

$$|a_j| \leq K(k, m, \mathcal{K}) \left(\frac{m}{\rho}\right)^j .$$

This concludes the proof. □

4. Technical lemmas

The aim of this section is to prove Lemma 3.2. We first recall a few classical points.

The algebraic curve of a polynomial $S \in \mathbb{C}[z, w]$ is the zero-set

$$\mathtt{R}_S := \{(z, w) \in \mathbb{C}^2 \,:\, S(z, w) = 0\} \,.$$

and one has the following standard result

Lemma 4.1. *For any integer $k \geq 1$ and for any polynomial $S \in \mathcal{P}(k)$, there exists a set $\mathcal{N}_S \subset \mathbb{C}$ (defined explicitly in Appendix A, see A.1) satisfying card $\mathcal{N}_S \leq \mathtt{N}_k$ - where $\mathtt{N}_k \in \mathbb{N}$ is an upper bound depending only of k - and such that over any simply connected domain $\mathtt{D} \subset \mathbb{C}$ the intersection of the algebraic curve $\mathtt{R}_S$ with $\mathtt{D} \times \mathbb{C}$ is the union of at most k disjoint graphs of holomorphic functions defined over $\mathtt{D}$ if and only if $\mathtt{D} \cap \mathcal{N}_S = \emptyset$.*

The proof of this result can be found by putting together known results on algebraic curves (see e.g. [30]). For the sake of clarity, it is given in appendix A.

Remark 4.1. Following ref. [32], the elements of $\mathcal{N}_S$ are called *excluded points*.

Remark 4.2. The number of graphs in Lemma 4.1 may be equal to zero, for example if $S(z, w) = z$, we have $\mathtt{R}_S = \{(z, w) \in \mathbb{C}^2 \,:\, z = 0\}$ and the point $z = 0$ is excluded by construction (see Appendix A).

Definition 4.1 (Riemann branches and leaves)**.** In the setting of Lemma 4.1, if $\mathtt{R}_S$ is non-empty over $\mathtt{D}$, the holomorphic functions whose graphs cover $\mathtt{D}$ are algebraic since their graphs solve the equation $S(z, w) = 0$ for all $z \in \mathtt{D}$. These functions will henceforth be called the *Riemann branches* of S over $\mathtt{D}$, whereas their graphs will be referred to as the *Riemann leaves* of S over $\mathtt{D}$.

It is a standard fact that, up to constant multiplicative factors, any polynomial $S \in \mathcal{P}(k)$ can be uniquely factorized as

$$S(z, w) = q(z)\, \Pi_{i=1}^{m} (S_i(z, w))^{j_i} \tag{4.1}$$

for some $1 \leq j_i \leq k$, $1 \leq m \leq k$, where the S_i's are non-constant, irreducible, mutually non-proportional polynomials. Hence, without any loss of generality, we can pass to the unit sphere in $\mathcal{Q}(k)$ and assume $||q|| = 1$ for an arbitrary norm $|| \cdot ||$.

We denote

$$\overline{S}(z, w) := \Pi_{i=1}^{m} (S_i(z, w))^{j_i} \tag{4.2}$$

and we have the polynomial product:

$$S(z, w) = q(z)\overline{S}(z, w). \tag{4.3}$$

We start by giving the following

Definition 4.2. $\mathcal{B} = \mathcal{B}(k, \rho) \subset \mathcal{P}(k)$ denotes the set of polynomials $S \in \mathcal{P}(k)\backslash\{0\}$ such that the polynomial $\overline{S}$ in decomposition (4.3) has no excluded points (definition A.1) in $\mathcal{D}_\rho(0)$.

Remark 4.3. Given $S \in \mathcal{B}$, by decomposition (4.3) and Definition A.1, the only possible excluded points for S in $\mathcal{D}_\rho(0)$ are those at which $q(z) = 0$. Inside the disk $\mathcal{D}_\rho(0)$, the algebraic curve $\mathtt{R}_S$ is therefore the union of at most k elements that can be either disjoint holomorphic Riemann leaves of $\overline{S}$ or vertical lines in $\mathbb{C}^2$ of the kind $z = z_0$, with $q(z_0) = 0$. In particular, all the Riemann branches of $S \in \mathcal{B}$ are holomorphic over $\mathcal{D}_\rho(0)$.

Remark 4.4. The set $\mathcal{A}$ of Definition 3.1 is contained in $\mathcal{B}$ and, with the notations of Theorem 1.2, the functions in $\mathcal{V}(k, \rho)$ are precisely those associated to the polynomials in $\mathcal{B}$.

In order to prove Lemma 3.2, we need the following

Lemma 4.2. $\mathcal{B} \cup \{0\}$ *is closed in* $\mathcal{P}(k)$*.*

The proof of Lemma 4.2 relies on arguments centered around Montel's theorem and is quite technical. It requires some intermediate results, which are stated in the sequel.

We start by considering a sequence $\{S_n(z, w)\}_{n\in\mathbb{N}}$ of polynomials in $\mathcal{B}\cup\{0\}$, converging to a polynomial $S \in \mathcal{P}(k)$. We can assume that $S \not\equiv 0$ otherwise there is nothing to prove; hence we have $S_n \not\equiv 0$ for n large enough.

Following decomposition (4.3), we write $S_n(z, w) := q_n(z)\overline{S}_n(z, w)$ and, by construction, the sequence of polynomials $\{q_n\}_{n\in\mathbb{N}}$ is in the compact unit sphere and admits a convergent subsequence. With slight abuse of notation, in the sequel we shall indicate this subsequence with the same symbol $\{q_n\}_{n\in\mathbb{N}}$ and we shall denote by $\widehat{q}$ its limit, which is not identically null by construction.

We recall that $\mathcal{N}_S$ and $\mathcal{N}_{S_n}$ (for $n \in \mathbb{N}$) denote the sets of excluded points of S and S_n, respectively. For $r > 0$ small enough, we remove from $\mathcal{D}_\rho(0)$ all open neighborhoods of radius r around the excluded points of S and consider the following compact set:

$$\mathtt{E}_r := \{z \in \overline{\mathcal{D}}_{\rho-r}(0) \;/\; |z - z_0| \geq r \text{ for } z_0 \in \mathcal{N}_S\} \subset \mathcal{D}_\rho(0). \tag{4.4}$$

Lemma 4.3. *There exists* $r_0 = r_0(\rho, k)$ *such that, for any* $0 < r \leq r_0$*, one has* $\mathtt{E}_r \neq \emptyset$ *and there exists an integer* $n_0 = n_0(r)$ *such that:*

$$\mathtt{E}_r \cap \mathcal{N}_{S_n} = \emptyset \textit{ for all } n \geq n_0\ . \tag{4.5}$$

Proof. The fact that $\mathtt{E}_r \neq \emptyset$ for r sufficiently small is an immediate consequence of Definition 4.4 and of the fact that card $\mathcal{N}_S$ is bounded by a number depending only on k (see Lemma 4.1).

As for the second part of the statement, since $S_n \longrightarrow S \in \mathcal{P}(k)$, and $q_n \to \widehat{q} \not\equiv 0$, there exists a polynomial $\widehat{S} \in \mathcal{P}(k)$ such that

$$\lim_{n\longrightarrow+\infty} S_n(z, w) = \lim_{n\longrightarrow+\infty} \overline{S}_n(z, w) \times \lim_{n\longrightarrow+\infty} q_n(z) = \widehat{S}(z, w) \times \widehat{q}(z). \tag{4.6}$$

By applying again decomposition (4.3) to $\widehat{S}$ we obtain $\widehat{S}(z, w) = \widetilde{q}(z)\overline{S}(z, w)$, so that we can write $S(z, w) = q(z)\overline{S}(z, w)$ by setting

$$q(z) := \widehat{q}(z) \times \widetilde{q}(z)\ . \tag{4.7}$$

Therefore, all the roots of $\widehat{q}$ are also roots of q and belong to $\mathcal{N}_S$. By construction (see also remark 4.3), all points in $\mathcal{N}_{S_n}$ are roots of $q_n(z) = 0$. Since $q_n \longrightarrow \widehat{q}$, taking into account the continuous dependence of the roots of a polynomial on its coefficients expressed in Theorem B.1, one has that for sufficiently high n the roots of q_n must be either r-close to the roots of $\widehat{q}$, and hence to some point of $\mathcal{N}_S$, or outside of the disc of radius $\mathcal{D}_{1/r}(0)$. Taking $r_0 < 1/\rho$, one has $\mathcal{D}_{1/r}(0) \supset \mathcal{D}_\rho(0)$, whence the conclusion. $\square$

We fix $0 < r \leq r_0$, with r_0 the bound in Lemma 4.3, and we consider a point $z^\star \in \mathtt{E}_r$, hence $z^\star$ is not an excluded point of S and any solution of $S^{z^\star}(w) := S(z^\star, w) = 0$ must belong to the image of a Riemann branch of S holomorphic in a neighbourhood of $z^\star$. We fix one of these branches and denote it with h. The continuous dependence of the zeros of a polynomial on its coefficients (Theorem B.1) ensures the existence of a sequence $\{w_n^\star\}_{n\in\mathbb{N}}$ of roots of $S_n^{z^\star}(w) := S_n(z^\star, w)$ such that

$$w_n^\star \longrightarrow h(z^\star)\ .$$

Lemma 4.3 and Remark 4.3 together with the fact that $S_n \in \mathcal{B}$ for all $n \in \mathbb{N}$ ensure that, for any fixed $n > n_0(r)$, the point $(z^\star, w_n^\star)$ must belong to the Riemann leaf of one of the branches of $\overline{S}_n$, denoted h_n, which is holomorphic over $\mathcal{D}_\rho(0)$. Hence we have the pointwise convergence

$$h_n(z^\star) \longrightarrow h(z^\star)\ . \tag{4.8}$$

We show in the sequel that the sequence $\{h_n\}_{n\in\mathbb{N}}$ admits a subsequence that converges uniformly on any compact subset of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a holomorphic function which extends h over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$. In order to prove this claim, which is fundamental to the proof of Lemma 4.2, we need the following results.

Lemma 4.4. *The Riemann branches of S are bounded on the compact sets included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. By construction, any point $\widehat{z} \in \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ is regular for S, hence there exists an open neighbourhood $V \subset \mathbb{C}$ of $\widehat{z}$ such that the algebraic curve $\mathtt{R}_S \cap \{V \times \mathbb{C}\}$ is composed of at most k graphs of holomorphic functions bounded over V. Since any compact set included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ can be covered by a finite number of these neighbourhoods, the claim is proved. $\square$

Lemma 4.5. *The sequence $\{h_n\}_{n\in\mathbb{N}}$ is locally bounded over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. If, by contradiction, there exists a compact $\mathtt{K} \subset \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ such that $\{h_n\}_{n\in\mathbb{N}}$ is unbounded over $\mathtt{K}$, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathtt{K}$ and a strictly increasing function φ over $\mathbb{N}$ such that the subsequence $\{|h_{\varphi(n)}(z_n)|\}_{n\in\mathbb{N}}$ diverges.

By Definition (4.4), $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S = \cup_{r>0}\,\mathtt{E}_r$, so there exists $0 < r \leq r_0$ small enough such that $\mathtt{K} \subset \mathtt{E}_r \subset \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$. Moreover, $\mathtt{E}_r$ is a compact, arc-connected set since it is $\overline{\mathcal{D}}_{\rho-r}(0)$ without a finite number of open disks. Then, for any $n \in \mathbb{N}$ it is always possible to construct a continuous arc:

$$\gamma_n : [0,1] \longrightarrow \mathtt{E}_r \text{ with } \gamma_n(0) = z^\star \text{ and } \gamma_n(1) = z_n .$$

We introduce the continuous functions:

$$\psi_n : [0,1] \longrightarrow \mathbb{R} \quad , \qquad \psi_n(t) := |h_{\varphi(n)}(\gamma_n(t))|\ .$$

Since $S_n^z \to S^z$ uniformly for $z \in \overline{\mathcal{D}}_{\rho-r}(0)$, Theorem B.1 ensures that, for all $\varepsilon > 0$, there exists $\mathtt{n}(\varepsilon) \in \mathbb{N}$ such that for all $n > \mathtt{n}(\varepsilon)$ and all $z \in \overline{\mathcal{D}}_{\rho-r}(0)$, the roots of S_n^z are either ε-close to the roots of S^z or in the complement of the closed disk $\overline{\mathcal{D}}_{1/\varepsilon}(0)$. Moreover, Lemma

4.4 ensures that the roots of S^z are uniformly bounded for all $z \in \mathtt{E}_r$. We indicate with $w_{max}(r)$ the maximal module that the Riemann branches of S can reach on $\mathtt{E}_r$ and we set

$$\varepsilon_0(r) = \frac{1}{w_{max}(r)+1}.$$

In this setting, we can consider a fixed integer $n > \mathtt{n}(\varepsilon_0(r))$ such that:

$$\psi_n(1) > \frac{1}{\varepsilon_0} = w_{max}(r) + 1 \tag{4.9}$$

and taking (4.8) into account, we also assume that n is high enough to ensure:

$$|\,\psi_n(0) - |h(z^\star)|\,| = |\,|h_{\varphi(n)}(z^\star)| - |h(z^\star)|\,| \leq |h_{\varphi(n)}(z^\star) - h(z^\star)| < \varepsilon_0 < 1 \tag{4.10}$$

hence $\psi_n(0) < |h(z^\star)| + 1 \leq w_{max}(r) + 1$.

With (4.9) and (4.10), the intermediate value theorem applied to ψ_n implies that there exists $t_\star \in]0,1[$ satisfying

$$\psi_n(t_\star) = w_{max}(r) + 1\ . \tag{4.11}$$

But $n > \mathtt{n}(\varepsilon_0(r))$ and $\gamma_n(t_\star) \in \overline{\mathcal{D}}_{\rho-r}(0)$, hence $h_{\varphi(n)}(\gamma_n(t_\star))$ is either in the complement of the closed disk $\overline{\mathcal{D}}_{1/\varepsilon_0}(0)$ and

$$\psi_n(t_\star) > \frac{1}{\varepsilon_0} = w_{max}(r) + 1 \tag{4.12}$$

or ε_0-close to the roots of S^z and

$$\psi_n(t_\star) < w_{max}(r) + \varepsilon_0 < w_{max}(r) + 1. \tag{4.13}$$

Conditions (4.11), (4.12) and (4.13) are in contradiction and the statement is proved. □

Lemma 4.6. *There exists a subsequence of $\{h_n\}_{n\in\mathbb{N}}$ which converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a Riemann branch of S (still denoted h) extending holomorphically h over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. With Lemma 4.5 and Montel's Theorem, it is possible to extract a subsequence - still denoted $\{h_n\}_{n\in\mathbb{N}}$ with slight abuse of notation - that converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a function holomorphic over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ which is also still denoted h. Finally, thanks to Lemma B.1 and to the fact that $S_n \longrightarrow S$, one has $S(z,h(z)) = 0$ for any $z \in \mathcal{D}_\rho(0)$. □

Remark 4.5. By the above Lemma, $\mathcal{N}_S$ does not contain any ramification points.

With the help of Lemma 4.6, we are now able to prove Lemma 4.2.

Proof. (Lemma 4.2)

The aim is to prove that the set of excluded points $\mathcal{N}_{\overline{S}}$ for $\overline{S}$ associated to the limit polynomial S is empty, from which the conclusion follows.

Assume that $z_0 \in \mathcal{N}_{\overline{S}}$. Since $\mathcal{N}_{\overline{S}}$ is a finite set, for $t > 0$ small enough the punctured disc $\dot{\mathcal{D}}_t(z_0) := \{z \in \mathcal{D}_\rho(0) : 0 < |z - z_0| < t\}$ is included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_{\overline{S}}$ and any branch h of the polynomial $\overline{S}$ is holomorphic in $\dot{\mathcal{D}}_t(z_0)$. Then, by Laurent's Theorem and by Proposition B.1, z_0 is either a removable singularity or a pole. We show that the second possibility does not occur.

If by contradiction z_0 is a pole for h, then $\lim_{z\longrightarrow z_0} h(z)$ is infinite and one can choose the radius t small enough so that $h(z)\neq 0$ for all $z\in \dot{\mathcal{D}}_t(z_0)$. Hence the function $\phi := 1/h$ is analytic on the punctured disc $\dot{\mathcal{D}}_t(z_0)$ and it is also bounded since its limit is zero when z goes to z_0. By Riemann's Theorem on removable singularities, ϕ admits a holomorphic extension, still denoted ϕ, on the whole disc $\mathcal{D}_t(z_0)$ satisfying $\phi(z_0)=0$.

Lemma 4.6 ensures that there exists a subsequence $\{h_{n_j}\}_{j\in\mathbb{N}}$ of Riemann branches for $\mathcal{S}_{n_j}$ (actually, by Remark 4.3, the branches h_{n_j} are analytic over $\mathcal{D}_\rho(0)$ since $\mathcal{S}_{n_j}\in\mathcal{B}$) which converges uniformly to h on the compact subsets of $\dot{\mathcal{D}}_t(z_0)\subset\mathcal{D}_\rho(0)\backslash\mathcal{N}_{\mathcal{S}}$. Consequently, the functions h_{n_j} do not vanish on any compact subset of the disk $\mathcal{D}_t(z_0)$ for j large enough. This ensures that the functions $\{\phi_{n_j}\}_{j\in\mathbb{N}} := \{1/h_{n_j}\}_{j\in\mathbb{N}}$ are holomorphic on $\mathcal{D}_t(z_0)$. Moreover, the sequence $\{\phi_{n_j}\}_{j\in\mathbb{N}}$ converges locally uniformly to ϕ on $\dot{\mathcal{D}}_t(z_0)$. Since both ϕ_{n_j} and ϕ are holomorphic at z_0, by the Maximum Principle, this convergence is actually locally uniform over the whole disc $\mathcal{D}_t(z_0)$.

On the one hand, we have $\phi(z_0)=0$ and $\phi(z)\neq 0$ for $z\in\dot{\mathcal{D}}_t(z_0)$, since in this domain $\phi(z)=1/h(z)$ and h is holomorphic on $\dot{\mathcal{D}}_t(z_0)$.

On the other hand, the terms of the subsequence $\{\phi_{n_j}\}_{j\in\mathbb{N}} := \{1/h_{n_j}\}_{j\in\mathbb{N}}$ are nowhere-vanishing on $\mathcal{D}_t(z_0)$ and ϕ_{n_j} is holomorphic in that domain. Consequently, by Hurwitz's Theorem on sequences of holomorphic functions, ϕ must be either identically zero or nowhere null on $\mathcal{D}_t(z_0)$.

We have obtained a contradiction and therefore $\lim_{z\longrightarrow z_0} h(z)$ is finite. By applying once again Riemann's Theorem on removable singularities, h admits an analytic extension $\widetilde{h}$ to the whole disc $\mathcal{D}_t(z_0)$. Moreover, $\widetilde{h}$ is a Riemann branch of $\overline{\mathcal{S}}$ in the whole disc $\mathcal{D}_t(z_0)$, since

$$\overline{\mathcal{S}}(z_0,\widetilde{h}(z_0)) = \lim_{z\longrightarrow z_0}\overline{\mathcal{S}}(z,h(z)) = 0\ .$$

It remains to rule out the possibility that z_0 is singular because the graphs of two distinct branches h and ℓ of the limit polynomial $\overline{\mathcal{S}}$ intersect on it. Assume that $h(z_0)=\ell(z_0)$. By Lemma 4.6 and by the previous arguments, there exist two subsequences $\{h_{n_j}\}_{j\in\mathbb{N}}$ and $\{\ell_{n_j}\}_{j\in\mathbb{N}}$ of branches associated to $\{\overline{\mathcal{S}}_{n_j}\}_{j\in\mathbb{N}}$ that approach respectively h and ℓ locally uniformly over $\mathcal{D}_t(z_0)$.

We first notice that h_{n_j} is distinct from ℓ_{n_j} for j large enough, otherwise there exists a subsequence of common branches $h_{n_j}=\ell_{n_j}$ for $\overline{\mathcal{S}}_{n_j}$ up to infinity which converges locally uniformly in $\mathcal{D}_t(z_0)$ respectively to h and ℓ. Consequently $h=\ell$ over $\mathcal{D}_t(z_0)$, which contradicts the assumption that h and ℓ are distinct. Moreover, since $\mathtt{R}_{\overline{\mathcal{S}}_{n_j}}$ is composed of distinct regular leaves over $\mathcal{D}_t(z_0)$ for any $j\in\mathbb{N}$, the functions $h_{n_j}-\ell_{n_j}$ never vanish over $\mathcal{D}_t(z_0)$.

Consequently, Hurwitz's theorem ensures that the sequence of holomorphic functions $\{h_{n_j}-\ell_{n_j}\}_{j\in\mathbb{N}}$ converges to a limit which either never vanishes or is identically zero over $\mathcal{D}_t(z_0)$. Here $\lim_{j\to+\infty}(h_{n_j}-\ell_{n_j})(z_0)=(h-\ell)(z_0)=0$, so $h=\ell$ everywhere over $\mathcal{D}_t(z_0)$, which is again in contradiction with the assumption that h and ℓ are distinct.

Therefore, we have proved that the algebraic curve of the limit polynomial $\overline{\mathcal{S}}$ is composed of disjoint and regular Riemann leaves over a neighborhood of z_0. Since the above arguments hold for any $z_0\in\mathcal{N}_{\mathcal{S}}$, the algebraic curve $\mathtt{R}_{\overline{\mathcal{S}}}$ is composed of distinct leaves over

$\mathcal{D}_\rho(0)$ and the branches of $\overline{S}$ are globally holomorphic over $\mathcal{D}_\rho(0)$, consequently $\mathcal{N}_{\overline{S}} = \emptyset$ (see Remark 4.3). $\square$

Lemma 4.2 is the cornerstone for the proof of Lemma 3.2.

Proof. (Lemma 3.2)

We start by proving the closure of $\mathcal{A} \cup \{0\}$ in $\mathcal{P}(k)$ and consider a sequence $\{S_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}\cup\{0\}$ which converges to a limit $S \in \mathcal{P}(k)$. One has $S \in \mathcal{B}\cup\{0\}$, since $\mathcal{A}\cup\{0\} \subset \mathcal{B}\cup\{0\}$ and $\mathcal{B} \cup \{0\}$ is closed by Lemma 4.2.

By hypothesis, for any fixed $n \in \mathbb{N}$ there exists a Riemann branch $g_n(z)$ which is analytic on $\mathcal{D}_\rho(0)$ and satisfies

$$S_n(z, g_n(z)) = 0\ , \quad g_n(0) = 0\ , \quad \max_{z\in\mathcal{K}}|g_n(z)| = 1\ . \tag{4.14}$$

If $S \equiv 0$ there is nothing to prove.

If $S \not\equiv 0$, we claim that $\{g_n\}_{n\in\mathbb{N}}$ has a subsequence that converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)$ to a branch g_S of S having the desired properties.

In fact, since $S \in \mathcal{B} \cup \{0\}$, the elements of the set $\mathcal{N}_S$ are the roots of $q(z) = 0$. Consequently, card $\mathcal{N}_S \leq k$ and card $\mathcal{K} > k$ ensures that there exists $z^\star \in \mathcal{K}\backslash\mathcal{N}_S$ such that $\{g_n(z^\star)\}_{n\in\mathbb{N}}$ is bounded. Up to the extraction of a subsequence, $g_n(z^\star)$ converges to a complex value $w^\star$. Moreover, since $z^\star$ is not an excluded point of $S \in \mathcal{B}$, we can ensure that $(z^\star, w^\star)$ belongs to a Riemann leaf of $\mathtt{R}_{\overline{S}}$ which is associated to a holomorphic Riemann branch over $\mathcal{D}_\rho(0)$, denoted g_S.

By the same arguments used in the proofs of Lemmas 4.5 and 4.6, the sequence $\{g_n\}_{n\in\mathbb{N}}$ admits a subsequence which converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a Riemann branch f_S associated to S. The holomorphy of f_S over $\mathcal{D}_\rho(0)$ (since $S \in \mathcal{B}$) and the Maximum Principle imply that the convergence is actually locally uniform on the whole set $\mathcal{D}_\rho(0)$. Then, by the uniqueness of the limit, we have $g_S(z^\star) = f_S(z^\star)$, which implies $g_S \equiv f_S$ over $\mathcal{D}_\rho(0)$ because $S \in \mathcal{B}$. This yields $\max_{\mathcal{K}} |g_S| = 1$ and $g_S(0) = 0$, hence g_S meets the requirements of Definition 3.1 and $S \in \mathcal{A}$.

Finally, it remains to prove that the function μ_Ω in Lemma 3.2 is continuous. Since $\{g_n\}_{n\in\mathbb{N}}$ converges locally uniformly to g_S in $\mathcal{D}_\rho(0)$, we can write

$$\lim_{n\longrightarrow+\infty} \left| \max_{z\in\mathcal{K}'} |g_S(z)| - \max_{z\in\mathcal{K}'} |g_n(z)| \right| \leq \lim_{n\longrightarrow+\infty} \left(\max_{z\in\mathcal{K}'} |g_S(z) - g_n(z)| \right) = 0\ , \tag{4.15}$$

for any compact $\mathcal{K}' \subset \mathcal{D}_\rho(0)$. By taking $\mathcal{K}' \equiv \overline{\Omega} \subset \mathcal{D}_\rho(0)$, we have

$$\mu_\Omega(S) := \max_{z\in\overline{\Omega}} |g_S(z)| = \lim_{n\longrightarrow+\infty} \left(\max_{z\in\overline{\Omega}} \left|g_n(z)\right| \right) =: \lim_{n\longrightarrow+\infty} \Big(\mu_\Omega(S_n) \Big)\ ,$$

which implies that μ_Ω is continuous. This concludes the proof of Lemma 3.2.

$\square$

Appendix A. Proof of Lemma 4.1

We start by stating two standard results of algebraic geometry.

Lemma A.1. *For any couple of positive integers k_1, k_2 consider two non-zero irreducible, non-proportional polynomials $Q_1 \in \mathcal{P}(k_1)$ and $Q_2 \in \mathcal{P}(k_2)$. Then the system $Q_1(z,w) = Q_2(z,w) = 0$ has at most $k_1 \times k_2$ solutions.*

Lemma A.2. *For $k \geq 2$, let $Q(z,w) \in \mathcal{P}(k)$ be an irreducible polynomial. Then*

$$card\,\{z \in \mathbb{C} \mid \exists w \in \mathbb{C} \,:\, Q(z,w) = \partial_w Q(z,w) = 0\} \leq k\;.$$

The first Lemma is a simple corollary of Bézout's Theorem (see e.g. [29], Th. 3.4a), while the second Lemma is also known (see e.g. Proposition 1 and its proof in [30]).

With these tools, we can now give the proof of Lemma 4.1.

Proof. The lemma is trivial if S depends only on w since we have $\mathcal{N}_S = \varnothing$ in this case because $\mathtt{R}_S$ is composed of a finite number of Riemann branches which are horizontal lines over the z-axis.

If $S \in \mathcal{P}(k)$ depends only on z, then $\mathtt{R}_S = \{(z,w) \in \mathbb{C}^2 \,:\, z = z_0,$ with $S(z_0) = 0\}$ and the thesis holds true since there are only vertical lines at the distinct roots of S (whose number is bounded by k) and no Riemann branches.

Let's now examine the case in which S depends on both variables where, up to multiplication by constant factors, any polynomial $S \in \mathcal{P}(k)$ can be factorized uniquely as

$$S(z,w) = q(z)\,\Pi_{i=1}^{m}(S_i(z,w))^{j_i} \tag{A.1}$$

for some $1 \leq j_i \leq k$, $1 \leq m \leq k$ and the S_i are non-constant, irreducible, mutually non-proportional polynomials.

We denote

$$\overline{S}(z,w) = \Pi_{i=1}^{m}(S_i(z,w))^{j_i} \text{ and } \widetilde{S}(z,w) = \Pi_{i=1}^{m} S_i(z,w) \tag{A.2}$$

and $\overline{S}^z(w) := \overline{S}(z,w)$, $\widetilde{S}^z(w) := \widetilde{S}(z,w)$ hence $\widetilde{S}^z \in \mathbb{C}[z][w]$ - with $\deg(\widetilde{S}^z) = \ell$ $\ell \in \{1,\ldots,k\}$ - and $a_\ell(z)$ is the corresponding leading coefficient.

We notice that decomposition A.1 and definition A.2 ensure that $\mathtt{R}_S$ is the union of the vertical lines $z = z^*$ with $q(z^*) = 0$ and of the Riemann surface $\mathtt{R}_{\overline{S}}$, moreover the Riemann surfaces $\mathtt{R}_{\overline{S}}$ and $\mathtt{R}_{\widetilde{S}}$ are identical.

Definition A.1 (Excluded points)**.** Taking decomposition A.1 into account, we define $\mathcal{N}_S \subset \mathbb{C}$ as the set of those points $z_0 \in \mathbb{C}$ that satisfy at least one of the following conditions

(1)
$$q(z_0) = 0 \qquad \text{(Vertical lines)}$$

(2) There exists $w_0 \in \mathbb{C}$ such that for some $i \in \{1,...,m\}$
$$\begin{cases} S_i(z_0,w_0) = 0 \\ \partial_w S_i(z_0,w_0) = 0 \end{cases} \qquad \text{(Ramification points)}$$

(3) There exists $w_0 \in \mathbb{C}$ such that for some $i,j \in \{1,...,m\}, i \neq j$
$$\begin{cases} S_i(z_0,w_0) = 0 \\ S_j(z_0,w_0) = 0 \end{cases} \qquad \text{(Intersection of graphs)}$$

(4)
$$a_\ell(z_0) = 0 \qquad \text{(Poles)}$$

Henceforth, we prove that over $\mathbb{C}\backslash\mathcal{N}_S$ the conclusions of Lemma 4.1 are valid and that we can choose the set $\mathcal{N}_S$ as the excluded points for S.

To see this, we fix a point $z^* \in \mathbb{C}\backslash\mathcal{N}_S$.

By negation of condition (1), we have $q(z) \neq 0$ in the vicinity of z^*, hence the vertical lines are excluded from the algebraic curve $\mathtt{R}_S$ in the vicinity of z^*.

Then, we also notice that for any value w^* such that $\overline{S}^{z^*}(w^*) = 0$, by decomposition (A.1) and negation of condition (3), one must have $S_i(z^*, w^*) = 0$, for exactly one $i \in \{1, ..., m\}$. Hence, by negation of condition (2) at (z^*, w^*), we can apply the implicit function theorem and there exists an open neighbourhood V around (z^*, w^*) such that $\mathtt{R}_S \cap V = \mathtt{R}_{\overline{S}} \cap V = \mathtt{R}_{\widetilde{S}} \cap V = \mathtt{R}_{S_i} \cap V$ is the graph of an unique holomorphic function.

Finally, the negation of condition (4) in a neighbourhood of z^* ensures that the polynomial $\widetilde{S}^z$ admits $\ell \leq k$ complex roots counted with multiplicity for z in the vicinity of z^*. With $z^* \in \mathbb{C}\backslash\mathcal{N}_S$, a direct computation ensures that the discriminant of $\widetilde{S}^{z^*}$ is non-zero since $\widetilde{S}^{z^*}$ and its derivative cannot have common roots, hence $\widetilde{S}^z$ admits simple roots for z in the vicinity of z^*.

This implies the existence of a neighborhood V of $z^* \in \mathbb{C}\backslash\mathcal{N}_S$ such that the algebraic curve $\mathtt{R}_S \cap V \times \mathbb{C} = \mathtt{R}_{\widetilde{S}} \cap V \times \mathbb{C}$ is the union of exactly $\ell \leq k$ disjoint graphs of holomorphic branches.

Hence, over a simply connected domain $\mathtt{D} \subset \mathbb{C}\backslash\mathcal{N}_S$, branch cuts can be avoided and the Riemann surface $\mathtt{R}_S$ is the finite union of at most k disjoint graphs of holomorphic functions.

Conversely, consider a simply connected complex domain $\mathtt{D}$ such that $\mathtt{R}_S \cap \mathtt{D} \times \mathbb{C}$ is the finite union of $\ell \in \{1, \ldots, k\}$ disjoint graphs of functions $h_1(z), \ldots, h_\ell(z)$ holomorphic over $\mathtt{D}$.

For a fixed point $z^* \in \mathtt{D}$, the polynomial S^{z^*} admits ℓ roots, and decomposition A.1 ensures that we have $q(z^*) \neq 0$. Moreover, the discriminant of $\widetilde{S}^z$ might be zero only at a finite number of points (since the discriminant is itself a polynomial) but we always have ℓ roots for $\widetilde{S}^z$ with $z \in \mathtt{D}$ as a consequence of our assumption that the Riemann leaves are distinct. Hence, the roots are simple and the degree of $\widetilde{S}^z$ is constant equal to ℓ for all $z \in \mathtt{D}$. Consequently, the discriminant of $\widetilde{S}^z$ is non-zero and $a_\ell(z) \neq 0$ for all $z \in \mathtt{D}$.

Moreover, for any $i, j \in \{1, ..., m\}, i \neq j$ and for any $w \in \mathbb{C}$, we have either $S_i(z^*, w) \neq 0$ or $S_j(z^*, w) \neq 0$ otherwise two distincts Riemann leaves associated respectively to S_i and S_j would intersect.

Finally, we have the decomposition $\widetilde{S}(z, w) = a_\ell(z)(w - h_1(z)) \ldots (w - h_\ell(z))$ for $(z, w) \in \mathtt{D} \times \mathbb{C}$, and for $z \in \mathtt{D}$ we can check that S^z and its derivative cannot have common roots under our assumptions. Hence, for any $w \in \mathbb{C}$ and any $i \in \{1, ..., m\}$, we have either $S_i(z^*, w) \neq 0$ or $\partial_w S_i(z^*, w) \neq 0$.

Then, we prove that the cardinality of $\mathcal{N}_S$ is bounded by a quantity depending only on k.

Conditions (1) and (4) are polynomial equations of degree less than or equal to k (q factorizes all the terms in z and $a_\ell(z)$ is the coefficient of the term of highest degree in w), hence they have at most k solutions. By Lemma A.1, condition (2) is satisfied at most at k points. Since the index i in (2) can assume at most k values, this condition yields k^2

singularities. In the same way, Lemma A.2 says that condition (3) is satisfied at most at k^2 points. Since the indices i, j in condition (3) can each take at most k values and $i \neq j$, we get $k^2\binom{k}{2}$ solutions. The sum of the previous estimates yields a bound depending only on k. □

Appendix B. Tools of algebraic geometry and applications

B.1. **On the dependence of the roots of a polynomial on its coefficients.** It is a standard fact in the study of algebra that the roots of a monic complex polynomial of one variable depend continuously on its coefficients. The same result holds true for non-monic polynomials once one takes solutions at infinity into account by means of the compact identification of $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere. Without entering into too many details, we state the following result, whose proof can be found in [18].

Theorem B.1. *Let $P(w) = a_n w^n + a_{n-1} w^{n-1} + ... + a_0$ be a non-zero complex polynomial of degree $k \leq n$. Let $\xi_1, ..., \xi_r$ be its roots in $\mathbb{C}$ with $m_1, ..., m_r$ their respective multiplicities. Fix $\varepsilon > 0$ small enough and denote with $\mathcal{D}_\varepsilon(\xi_1), ..., \mathcal{D}_\varepsilon(\xi_r)$ the disjoint disks of radius ε centered at $\xi_1, ..., \xi_r$, such that $\mathcal{D}_\varepsilon(\xi_j) \subset \mathcal{D}_{1/\varepsilon}(0)$ for all $j \in \{1, ..., r\}$. Then, there exists $\delta(\varepsilon) > 0$ such that every complex polynomial $Q(w) = b_n w^n + b_{n-1} w^{n-1} + ... + b_0$ satisfying $|b_j - a_j| < \delta(\varepsilon)$ for all $j \in \{0, ..., n\}$ has m_i roots (counted with multiplicity) in each $\mathcal{D}_\varepsilon(\xi_i)$ for $i \in \{1, ..., r\}$ and* $\deg(Q) - k$ *roots belonging to the complement of $\mathcal{D}_{1/\varepsilon}(0)$.*

This theorem formalizes the intuitive idea that, if one takes a polynomial

$$Q(w) = a_n w^n + a_{n-1}, w^{n-1} + ... + a_0, \text{ with } a_n \neq 0$$

and makes $a_{k+1}, a_{k+2}, ..., a_n$ tend to zero while $a_k \neq 0$, then $n - k$ solutions "continuously go to infinity" and k solutions, counted with their multiplicities, "stay finite."

B.2. **Application to sequences of algebraic functions.**

Lemma B.1. *Take an open bounded set $\mathbb{U} \subset \mathbb{C}$, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic algebraic functions on $\mathbb{U}$ associated to polynomials of degree $k \in \mathbb{N}$ and converging in $\mathbb{U}$ to a holomorphic function g. Then g is an algebraic function. Moreover, there exists a sequence of polynomials $\{Q_n \in \mathcal{P}(k)\}_{n\in\mathbb{N}}$ solving the graphs of the functions in $\{g_n\}_{n\in\mathbb{N}}$ which converges to a polynomial $Q \in \mathbb{C}[z, w]$ that solves graph(g) everywhere in $\mathbb{U}$.*

Proof. For any $n \in \mathbb{N}$, the graph of the function g_n satisfies $S_n(z, g_n(z)) = 0$ for some polynomial $S_n \in \mathcal{P}(k)\backslash\{0\}$ and the equation $S_n(z, g_n(z)) = 0$ is invariant when S_n is multiplied by any non-zero constant. Without loss of generality, one can choose an arbitrary norm $||\cdot||$ in $\mathcal{P}(k) \simeq \mathbb{C}^m$, with $m = (k + 1)(k + 2)/2$, and consider the sequence of polynomials $\{Q_n\}_{n\in\mathbb{N}}$ associated to $\{g_n\}_{n\in\mathbb{N}}$ by defining, for any $n \in \mathbb{N}$:

$$Q_n(z, w) := \frac{S_n(z, w)}{||S_n||} \text{ with } Q_n(z, g_n(z)) = 0 \text{ and } Q_n \in \mathbb{S}^m \tag{B.1}$$

where $\mathbb{S}^m$ denotes the unitary sphere in $\mathcal{P}(k) \simeq \mathbb{C}^m$. By compactness of $\mathbb{S}^m$, there exists a subsequence $\{Q_{n_j}\}_{j\in\mathbb{N}}$ converging to a polynomial $Q \in \mathbb{S}^m$. Moreover, if we denote

$Q^z_{n_j}(w) := Q_{n_j}(z,w)$ and $Q^z(w) := Q(z,w)$ - hence $Q^z_{n_j}$ and Q^z belong to $\mathcal{Q}(k)$ for any fixed $z \in \mathbb{C}$ - we have the convergence

$$\lim_{j \longrightarrow +\infty} ||Q^{z^*}_{n_j} - Q^{z^*}|| = 0 \tag{B.2}$$

for any fixed $z^* \in \mathtt{U}$ and with respect to any norm in $\mathcal{Q}(k)$.

The convergence in (B.2) and Theorem B.1 imply that the sequence $\{g_{n_j}(z^*)\}_{j\in\mathbb{N}}$ approaches a root of Q^{z^*} for any $z^* \in \mathtt{U}$. Since g_{n_j} converges over $\mathtt{U}$ to g, then $g(z)$ is a solution of $Q^z(w) = 0$ for any $z \in \mathtt{U}$.

Finally, since g is holomorphic over $\mathtt{U}$, it is a Riemann branch of $Q \in \mathcal{P}(k)$ over $\mathtt{U}$. □

B.3. **Non-existence of essential singularities for algebraic functions.**

Proposition B.1. *An algebraic function f cannot have any essential singularity.*

Proof. By Lemma 4.1 and decomposition A.1, the singularities of f are included in the set $\mathcal{N}_{\overline{S}}$ of excluded points (see A.1). Hence, suppose by contradiction that $z_0 \in \mathcal{N}_{\overline{S}}$ is an essential singularity. Since the cardinality of $\mathcal{N}_{\overline{S}}$ is finite, z_0 is isolated. Then, the Casorati-Weierstrass Theorem holds and, for any fixed $w_0 \in \mathbb{C}$, one can build a sequence $\{z_k\}_{k\in\mathbb{N}}$ converging to z_0 and such that

$$\lim_{k \longrightarrow +\infty} f(z_k) = w_0 \, .$$

But w_0 is also a root of the one-variable polynomial $\overline{S}^{z_0}(w) := \overline{S}(z_0, w)$ since $f(z)$ is a Riemann branch of $\overline{S}$ in a punctured neighborhood centered at z_0 and

$$\overline{S}^{z_0}(w_0) := \overline{S}(z_0, w_0) = \lim_{k \longrightarrow +\infty} \overline{S}(z_k, f(z_k)) = 0.$$

This construction holds for any $w_0 \in \mathbb{C}$ and the polynomial $\overline{S}^{z_0}$ is null but, necessarily, $(z - z_0)$ is a factor of $\overline{S}$ and this is not possible with decomposition A.1. Hence, $f(z)$ cannot have an essential singularity at z_0. □

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References

[1] V. Arnol'd. Instability of dynamical systems with several degrees of freedom. Soviet. Math. Dokl., 156(1):9–12, 1964.

[2] V. Arnol'd, V. Kozlov, A.I. Neishtadt, and E. Khukhro. Mathematical Aspects of Classical and Celestial Mechanics. Springer, 2010.

[3] S. Barbieri. On the algebraic properties of exponentially stable integrable hamiltonian systems. Annales de la Faculté des Sciences de Toulouse, 31(5):1365–1390, 2020.

[4] S. Barbieri, J.P. Marco, and J.E. Massetti. Analytic smoothing and Nekhoroshev estimates for Hölder steep Hamiltonians. Commun. Math. Phys., (396):349–381, 2022.

[5] G. Binyamini and D. Novikov. Complex cellular structures. Ann. of Math., 190:145–248, 2019.

[6] J. Bolte, A. Daniilidis, O. Ley, and L. Mazet. Characterizations of Łojasiewicz inequalities: subgradient flows, talweg, convexity. Trans. Amer. Math. Soc., 362:3319–3363, 2010.

[7] J. Bourgain, L. Goldstein, and W. Schlag. Anderson localization for Schrödinger operators on Z^2 with quasi-periodic potential. Acta Math., (188):41–86, 2002.

[8] M. Briskin and Y. Yomdin. Algebraic families of analytic functions. Journal of Differential Equations, 136(2):248–267, 1997.

[9] A. Brudnyi. Small amplitude limit cycles and the distribution of zeros of families of analytic functions. Ann. Math., 154(2):227–243, 2001.

[10] A. Brudnyi. Jensen inequality for a family of analytic functions and an estimate for the average number of limit cycles. Bull. Lond. Math. Soc., 35(2):229–235, 2003.

[11] A. Brudnyi. On local behavior of holomorphic functions along complex submanifolds of $\mathbb{C}^N$. Invent. Math., 173:315–373, 2008.

[12] A. Brudnyi. Bernstein type inequalities for restrictions of polynomials to complex submanifolds of $\mathbb{C}^n$. Journal of Approximation Theory, 225:106–147, 2018.

[13] D. Burguet. A proof of Yomdin-Gromov's algebraic lemma . Israel Journal of Mathematics, 168:291–316, 2008.

[14] L. Chierchia. Kolmogorov's 1954 paper on nearly-integrable Hamiltonian systems. Regular and Chaotic Dynamics, 13(2):130–139, 2008.

[15] R. Cluckers, J. Pila, and A. Wilkie. Uniform parameterization of subanalytic sets and diophantine applications. Ann. Sci. Éc. Norm. Super., 53:1–42, 2020.

[16] D. Coman and E.A. Poletsky. Transcendence measures and algebraic growth of entire functions. Invent. Math., 170:103–145, 2007.

[17] G. Comte and Y. Yomdin. Zeroes and rational points of analytic functions. Annales de l'Institut Fourier, (6):2445–2476, 2018.

[18] F. Cucker and A. Gonzalez-Corbalan. An alternate proof of the continuity of the roots of a polynomial. The American Mathematical Monthly, 96(4):342–345, 1989.

[19] C. Fefferman and R. Narasimhan. Bernstein's inequality and the resolution of spaces of analytic functions. Duke Math. J., 81(1):77–98, 1995.

[20] C. Fefferman and R. Narasimhan. A local Bernstein inequality on real algebraic varieties. Math Z., 223(4):673–692, 1996.

[21] J.P. Françoise. Poincaré–Andronov–Hopf Bifurcation and the Local Hilbert's 16th Problem. Qual. Theory Dyn. Syst., 11:61–77, 2012.

[22] J.P. Françoise and Y. Yomdin. Bernstein inequalities and applications to analytic geometry and differential equations. Journal of Functional Analysis, 146:185–205, 1997.

[23] O. Friedland and Y. Yomdin. (s,p)-valent functions. Lecture Notes in Math., 2169:123–136, 2017.

[24] M. Gromov. Entropy, homology and semialgebraic geometry. Astérisque, 145, 1987.

[25] M. Guzzo, L. Chierchia, and G. Benettin. The Steep Nekhoroshev's Theorem. Commun. Math. Phys., 342:569–601, 2016.

[26] Y. Ilyashenko. A criterion of steepness for analytic functions. Uspekhi Mat. Nauk., 41:193–194, 1986.

[27] Y. Ilyashenko. Centennial history of Hilbert's 16th problem. Bull. Am. Math. Soc., 39:301–354, 2002.

[28] Y. Ilyashenko and S. Yakovenko. Counting real zeros of analytic functions satisfying linear ordinary differential equations. Journal of Differential Equations, (126):87–105, 1996.

[29] K. Kendig. A guide to plane algebraic curves. Mathematical Association of America, 2012.

[30] N. Narasimhan. Compact Riemann Surfaces - Lectures in Mathematics ETH Zürich. Springer Basel AG, 1992.

[31] F. Nazarov, M. Sodin, and A. Volberg. Local dimension-free estimates for volumes of sublevel sets of analytic functions. Isr. J. Math., 133:269–283, 2002.

[32] N. N. Nekhoroshev. Stable lower estimates for smooth mappings and for gradients of smooth functions. Math USSR Sb., 19(3):425–467, 1973.

[33] N. N. Nekhoroshev. An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems. I. Russian Mathematical Surveys, 32(6):1–65, 1977.

[34] N. N. Nekhoroshev. An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems. II. Topics in Modern Mathematics, Petrovskii Seminar, (5):5–50, 1979.

[35] L. Niederman. Hamiltonian stability and subanalytic geometry. Annales de l'Institut Fourier, 56(3):795–813, 2006.

[36] R. Pierzchała. Remez-type inequality on sets with cusps. Adv. Math., 281:508–552, 2015.

[37] N. Roytwarf and Y. Yomdin. Bernstein Classes. Annales de l'Institut Fourier, 47(3):825–858, 1998.

[38] A. Sadullaev. An estimate for plynomials on analytic sets. Math. USSR Izvestiya, 20(3):493–502, 1983.

[39] R. Tijdeman. On the number of zeros of general exponential polynomials. Indag. Math., 74:1–7, 1971.

[40] Y. Yomdin. C^k-resolution of semialgebraic mappings. Addendum to: Volume growth and entropy. Israel J. Math., 57:301–317, 1987.

[41] Y. Yomdin. Analytic reparametrization of semi-algebraic sets. J. Complexity, 24:54–76, 2008.

[42] Y. Yomdin. Remez-type inequality for discrete sets. Israel J. Math., 186:45–60, 2011.

[43] Y. Yomdin. Smooth parametrizations in dynamics, analysis, diophantine and computational geometry. Japan Journal Ind. Appl. Math., 32:411–435, 2015.

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