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Stability in Hamiltonian Systems : steepness and regularity in Nekhoroshev theory

Stabilité des Systèmes Hamiltoniens : escarpement et régularité dans la théorie de Nekhoroshev

Thèse de doctorat de l'Université Paris-Saclay et de l'Università degli Studi Roma Tre

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Thèse soutenue à Paris-Saclay, le 31 Mai 2023, par

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Titre : Stabilité des Systèmes Hamiltoniens : escarpement et régularité dans la théorie de Nekhoroshev

Mots clés : Systèmes Hamiltoniens, Steepness, Théorie de Nekhoroshev, Théorie KAM, Géométrie semi-algébrique, Inégalité de Bernstein

Résumé : Cette thèse est consacrée à l'étude de la stabilité des solutions des systèmes Hamiltoniens presque intégrables (au sens d'Arnold-Liouville). Le premier axe porte sur la généricité de la propriété d'escarpement (steepness), une condition de transversalité sur le gradient, due à Nekhoroshev, qui assure la stabilité sur des temps très longs des solutions d'un système presque-intégrable suffisamment régulier. L'objectif dans cette partie est double : il s'agit d'une part de clarifier les méthodes de géométrie algébrique réelle et d'analyse complexe qui permettent de prouver la généricité de la propriété d'escarpement et, d'autre part, d'utiliser ces méthodes pour établir des critères explicites qui entraînent l'escarpement d'une fonction donnée, ce qui constitue un aspect important dans les applications de la théorie. Dans le deuxième axe de cette thèse, on développe de manière non-triviale un argument classique d'approximation analytique, qui permet de généraliser à la classe de régularité Hölder les estimations de stabilité de Nekhoroshev initialement valides pour des systèmes Hamitoniens presque intégrables analytiques. Une fois qu'une approximation analytique adaptée est construite, les estimations sont déduites de manière relativement rapide : de plus, cette technique permet d'étendre à une régularité plus faible les estimations de stabilité les plus fines prouvées en classe analytique. Enfin, on s'intéresse au problème de la stabilité en temps infini des systèmes Hamiltoniens presque-intégrables analytiques : il s'agit de généraliser les résultats fins sur la mesure des tores invariants obtenus avec la théorie KAM - prouvés pour une classe générique de systèmes mécaniques - au cas de systèmes associés à des Hamiltoniens plus généraux.

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Titolo : Stabilità dei sistemi Hamiltoniani : proprietà di ripidità e regolarità in Teoria di Nekhoroshev

Parole chiave : Sistemi Hamiltoniani, Steepness, Teoria di Nekhoroshev, Teoria KAM, Geometria semi-algebrica, Disuguaglianza di Bernstein

Sunto : Scopo di questa tesi è lo studio della stabilità delle soluzioni dei sistemi Hamiltoniani quasi integrabili (secondo Arnold-Liouville). La prima parte verte sulla genericità della proprietà di ripidità (steepness) : si tratta di una condizione di trasversalità sul gradiente, introdotta da Nekhoroshev, che garantisce la stabilità su tempi molto lunghi delle soluzioni di un sistema quasi integrabile sufficientemente regolare. L'obiettivo di questa prima sezione è duplice : da un lato vengono chiariti gli argomenti di geometria algebrica reale e di analisi complessa che permettono di provare la genericità della ripidità e, dall'altro, si utilizzano questi metodi per stabilire dei criteri espliciti che permettano di verificare la ripidità di una funzione data, un punto importante in vista di possibili applicazioni. La seconda parte della tesi riguarda l'estensione non triviale di alcuni risultati classici di approssimazione analitica : l'obiettivo è quello di generalizzare alla classe di regolarità Hölder le stime di stabilità di Nekhoroshev valide per sistemi Hamiltoniani quasi integrabili di classe analitica. Una volta ottenuti dei risultati di approssimazione analitica adatti, le stime di Nekhoroshev vengono dedotte in maniera relativamente rapida ; inoltre, tale tecnica permette di estendere i risultati più fini sulla stabilità dei sistemi quasi integrabili analitici a sistemi di regolarità più debole. Infine, nell'ultima parte viene esplorato il problema della stabilità in tempi infiniti dei sistemi Hamiltoniani quasi integrabili analitici : si tratta di generalizzare i risultati sulla misura dell'insieme complementare dei tori KAM invarianti - dimostrati nel caso di sistemi meccanici generici - a classi più ampie di sistemi quasi integrabili.

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Stability in Hamiltonian Systems: steepness and regularity in Nekhoroshev theory

Stabilité des Systèmes Hamiltoniens : escarpement et régularité dans la théorie de Nekhoroshev

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Ad Ivan Casaglia, che fin dai tempi del liceo mi ha fatto amare la matematica

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Paris, May 25th 2023

Abstract

This thesis is devoted to the study of the stability of the solutions of Hamiltonian dynamical systems which are close to integrable (in the sense of Arnold-Liouville). We firstly consider the genericity of functions satisfying the steepness property, a transversality condition on the gradient - introduced by Nekhoroshev - which ensures the stability over long timespans of the solutions of a smooth enough nearly-integrable system. The goal of this part is two-fold: on the one hand, we clarify the arguments of real-algebraic geometry and complex analysis that enter into the proof of the genericity of steepness; on the other hand, these techniques yield new explicit criteria that allow to check whether a given function is steep, which constitutes an important aspect in view of applications. The second axis of the thesis is centered around a non-trivial improvement of a classical result on analytic approximation: the aim in this case consists in extending to the Hölder case the classic Nekhoroshev's estimates of stability holding for generic, analytic, nearly-integrable systems. Once a suitable analytic approximation is constructed, estimates are obtained in a relatively effortless way. Moreover we extend to lower regularity the most accurate Nekhoroshev's estimates available in the analytic class. The final part of this thesis investigates some aspects of the stability in infinite-time of real-analytic nearly-integrable Hamiltonian systems: namely, by making use of quantitative results of Morse-Sard's Theory, we discuss the extension to more general Hamiltonians of the existing refined results about the Lebesgue measure of the complementary set of invariant KAM tori in generic mechanical systems.

Résumé

Cette thèse est consacrée à l'étude de la stabilité des solutions des systèmes Hamiltoniens presque intégrables (au sens d'Arnold-Liouville). Le premier axe porte sur la généricité de la propriété d'escarpement (steepness), une condition de transversalité sur le gradient, due à Nekhoroshev, qui assure la stabilité sur des temps très longs des solutions d'un système presque-intégrable suffisamment régulier. L'objectif dans cette partie est double : il s'agit d'une part de clarifier les méthodes de géométrie algébrique réelle et d'analyse complexe qui permettent de prouver la généricité de la propriété d'escarpement et, d'autre part, d'utiliser ces méthodes pour établir des critères explicites qui entraînent l'escarpement d'une fonction donnée, ce qui constitue un aspect important dans les applications de la théorie. Dans le deuxième axe de cette thèse, on développe de manière non-triviale un argument classique d'approximation analytique, qui permet de généraliser à la classe de régularité Hölder les estimations de stabilité de Nekhoroshev initialement valides pour des systèmes Hamitoniens presque intégrables analytiques. Une fois qu'une approximation analytique adaptée est construite, les estimations sont déduites de manière relativement rapide: de plus, cette technique permet d'étendre à une régularité plus faible les estimations de stabilité les plus fines prouvées

en classe analytique. Enfin, on s'intéresse au problème de la stabilité en temps infini des systèmes Hamiltoniens presque-intégrables analytiques: il s'agit de généraliser les résultats fins sur la mesure des tores invariants obtenus avec la théorie KAM - prouvés pour une classe générique de systèmes mécaniques - au cas de systèmes associés à des Hamiltoniens plus généraux.

Sunto

Scopo di questa tesi è lo studio della stabilità delle soluzioni dei sistemi Hamiltoniani quasi integrabili (secondo Arnold-Liouville). La prima parte verte sulla genericità della proprietà di ripidità (steepness): si tratta di una condizione di trasversalità sul gradiente, introdotta da Nekhoroshev, che garantisce la stabilità su tempi molto lunghi delle soluzioni di un sistema quasi integrabile sufficientemente regolare. L'obiettivo di questa prima sezione è duplice: da un lato vengono chiariti gli argomenti di geometria algebrica reale e di analisi complessa che permettono di provare la genericità della ripidità e, dall'altro, si utilizzano questi metodi per stabilire dei criteri espliciti che permettano di verificare la ripidità di una funzione data, un punto importante in vista di possibili applicazioni. La seconda parte della tesi riguarda l'estensione non triviale di alcuni risultati classici di approssimazione analitica: l'obiettivo è quello di generalizzare alla classe di regolarità Hölder le stime di stabilità di Nekhoroshev valide per sistemi Hamiltoniani quasi integrabili di classe analitica. Una volta ottenuti dei risultati di approssimazione analitica adatti, le stime di Nekhoroshev vengono dedotte in maniera relativamente rapida; inoltre, tale tecnica permette di estendere i risultati più fini sulla stabilità dei sistemi quasi integrabili analitici a sistemi di regolarità più debole. Infine, nell'ultima parte viene esplorato il problema della stabilità in tempi infiniti dei sistemi Hamiltoniani quasi integrabili analitici: si tratta di generalizzare i risultati sulla misura dell'insieme complementare dei tori KAM invarianti - dimostrati nel caso di sistemi meccanici generici - a classi più ampie di sistemi quasi integrabili.

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Chapter 1

Introduction générale

1.1 Le problème général

Le formalisme Hamiltonien est le cadre qui apparaît naturellement dans la description mathématique de systèmes fondamentaux issus de la physique: il présente beaucoup d'avantages que nous allons rappeler brièvement.

Les résultats suivants sont valables pour un système Hamiltonien quelconque mais nous allons exposer ces théorèmes dans le cas particulier le plus simple des systèmes mécaniques où la force dérive d'un potentiel et, plus généralement, dans le cas des systèmes lagrangiens globalement réguliers (voir [4]).

On étudie alors le mouvement d'un point sur une variété riemannienne $\mathcal{M}$ (variété de configuration): le système d'ordre un associé aux équations de la mécanique classique (i.e. : $\ddot{q} = -\partial U(q)$ pour l'espace euclidien usuel, où q désigne des coordonnées locales de $\mathcal{M}$) peut être transformé par dualité grâce à la transformation de Legendre. Il prend alors la forme canonique :

$$\dot{p} = -\partial_q H(p, q) \quad ; \qquad \dot{q} = \partial_p H(p, q) \tag{1.1.1}$$

où H est une fonction numérique différentiable sur le fibré cotangent $T^*\mathcal{M}$ et p sont les coordonnées conjuguées à q.

La fonction H est appelée *fonction Hamiltonienne*, ou plus simplement *Hamiltonien*.

Dans le cas de l'espace euclidien, le Hamiltonien prend la forme habituelle de l'énergie avec la somme de l'énergie cinétique $\frac{||p||^2}{2}$ et de l'énergie potentielle $U(q)$.

D'un point de vue plus géométrique, $T^*\mathcal{M}$ peut être muni d'une structure symplectique canonique grâce à la forme de Liouville $\omega = \sum_i dp_i \wedge dq_i$ où $(q_1, \dots, q_n)$ sont des coordonnées locales sur $\mathcal{M}$ et $(p_1, \dots, p_n)$ leurs coordonnées conjuguées (aussi appelées *impulsions*). L'équation (1.1.1) est ainsi celle qui est associée au gradient symplectique X_H^* de H défini par

$$i_{X_H}\omega := \omega(X_H, \cdot) = dH \ . \tag{1.1.2}$$

De plus, on dit qu'un difféomophisme (local) Φ du fibré cotangent $T^*\mathcal{M}$ est *symplectique* ou *canonique* s'il préserve la forme de Liouville, i.e. : $\Phi^*\omega = \omega$. Plus généralement, un difféomorphisme entre deux variétés symplectiques $(\mathcal{M}, \omega)$ et $(\tilde{\mathcal{M}}, \tilde{\omega})$ qui transporte ω sur $\tilde{\omega}$ (i.e. : $\Phi^*\omega = \tilde{\omega}$) est un difféomorphisme symplectique.

Si l'on considère un Hamiltonien H défini sur $T^*\mathcal{M}$ alors le flot Φ^t_H associé au système canonique gouverné par H est une transformation symplectique sur son domaine de définition (voir [89]). Un tel difféomorphisme conserve la forme canonique (1.1.1) des équations de Hamilton, c'est à dire que le système dans les nouvelles variables $(p, q) = \Phi(P, Q)$ est associé au gradient symplectique du Hamiltonien $K(P, Q) = H \circ \Phi(P, Q)$. Ceci est un avantage important du formalisme Hamiltonien puisque dans le système initial défini sur le fibré tangent TM, une transformation ne peut pas faire intervenir les positions q et les vitesses $\dot{q}$ tout en conservant la forme des équations étudiées, alors que dans le cadre canonique on peut mélanger les positions q et les impulsions p. Il s'agit d'un des ingrédients centraux dans la théorie des perturbations Hamiltoniennes, qui est à la base de la théorie de Nekhoroshev, dont on parlera par la suite.

1.2 Les systèmes intégrables

Un autre point remarquable apparaît dans l'étude des systèmes Hamiltoniens.

A priori, intégrer un système différentiel ordinaire de dimension $2n$ impose de déterminer $2n$ intégrales premières. Ici, l'existence de n intégrales premières peut permettre de garantir que le système (1.1.1) est intégrable par quadrature.

Remarquons d'abord qu'une fonction $F \in C^1(T^*\mathcal{M}, \mathbb{R})$ est constante le long des solutions $(p(t), q(t))$ du système associé à un Hamiltonien $H \in C^1(T^*\mathcal{M}, \mathbb{R})$ si et seulement si

$$\frac{d}{dt}(F(p(t), q(t)) = \partial_q F \partial_p H - \partial_p F \partial_q H = \{F, H\}(t) = 0 \tag{1.2.1}$$

sur le domaine de définition de la solution considérée.

La fonction $\{F, H\}$ s'appelle *le crochet de Poisson* de F et de H.

On peut alors énoncer le théorème d'Arnold-Liouville :

Theorem 1.2.1 (Arnold-Liouville)**.** *On considère un Hamiltonien $H \in C^1(T^*\mathcal{M}, \mathbb{R})$ dont le système associé admet n intégrales premières indépendantes en involution, que l'on note $\Psi_i \in C^1(T^*\mathcal{M}, \mathbb{R})$ pour $i \in \{1, \dots, n\}$, i.e. :*

$$\Psi_1 = H \quad ; \qquad \{\Psi_i, \Psi_j\} = 0 \ \text{pour}\ (i, j) \in \{1, \dots, n\}^2 \quad ; \qquad d\Psi_1 \wedge \dots \wedge d\Psi_n \neq 0.$$

Soit $\alpha \in \mathbb{R}^n$ tel que $\mathcal{N}_\alpha = \left\{(p, q) \in T^*\mathcal{M} \text{ avec } \left(\Psi_1, \dots, \Psi_n\right) = \alpha\right\}$ est non vide, compacte et connexe.

Alors, $\mathcal{N}_\alpha$ est difféomorphe au tore $\mathbb{T}^n$ de dimension n.

De plus, il existe un ouvert $V \subset T^*\mathcal{M}$ qui contient $\mathcal{N}_\alpha$, qui est invariant pour le flot Hamiltonien associé à H et qui est canoniquement difféomorphe à $U \times \mathbb{T}^n$ où U est un ouvert dans $\mathbb{R}^n$.

En effet, il existe un difféomophisme canonique Φ tel que le système (1.1.1) dans les nouvelles variables $(I, \theta) = \Phi^{-1}(p, q)$ est associé au hamiltonien $K(I) = H \circ \Phi(I, \theta)$ qui est indépendant des angles. Alors les équations associées sont trivialement intégrables puisque $\dot{I} = 0$ et $\dot{\theta} = \nabla K(I)$ où ∇K est le gradient de K sur U.

Les variables $(I, \theta) \in U \times \mathbb{T}^n$ sont appelées variables *actions-angles* du Hamiltonien intégrable K.

Géométriquement, les propriétés précédentes se traduisent par le fait que l'espace des phases est feuilleté en tores invariants de dimension n qui portent des solutions où les coordonnées angulaires varient linéairement (avec les fréquences ∇K): on dit alors que les solutions sont quasi-périodiques.

1.3 Trois exemples de systèmes intégrables

— Une chaine de *rotateurs* dont le Hamiltonien associé s'écrit

$$K(I) = \frac{1}{2}\left(I_1^2 + \ldots + I_n^2\right) \quad , \qquad (I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n \ :$$

il s'agit ici d'une collection de particules libres sur le tore $\mathbb{T}^n$ de dimension n.

— Une chaine d'*oscillateurs harmoniques* (i.e. : n ressorts découplés de raideurs k_i^2 pour i variant entre 1 et n) dont le Hamiltonien associé s'écrit

$$H(p, q) = \sum_{i=1}^{n} \frac{p_i^2 + \left(k_i q_i\right)^2}{2} \quad , \qquad (p, q) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Le passage en variables actions-angles s'obtient en utilisant les coordonnées polaires symplectiques :

$$p_i = \sqrt{2 k_i I_i}\cos(\theta_i) \quad ; \qquad q_i = \sqrt{2\frac{I_i}{k_i}}\sin(\theta_i) \quad \text{pour } i \in \{1, \ldots, n\} \ ,$$

ainsi les actions correspondent aux rayons des tores invariants et le Hamiltonien devient $K(I) = k_1 I_1 + \ldots + k_n I_n$, qui est intégrable.

C'est le même type de Hamiltonien qui apparait au voisinage d'une position d'équilibre elliptique (voir [32] et les références dans ce travail).

— Le *problème de Kepler* où l'on étudie le mouvement d'un point soumis à l'attraction gravitationnelle d'un corps fixe placé à l'origine.

Le Hamiltonien considéré est à trois degrés de liberté et s'écrit en coordonnées cartésiennes $H(p, q) = \frac{||p||^2}{2} - \frac{k}{||q||}$ avec une constante k positive.

Il est bien connu que pour des énergie négatives $H(p,q) < 0$, les solutions sont des ellipses admettant un foyer à l'origine. Les variables actions-angles (dites de Delaunay) s'expriment alors en fonction des éléments caractéristiques de l'ellipse parcouru.

Plus précisément, il s'agit de la longueur du demi-grand-axe, de l'excentricité, des trois angles d'Euler permettant de repérer la direction du demi-grand-axe dans l'espace par rapport à un axe de référence. Enfin, la seule variable évoluant rapidement est l'angle polaire du point considéré défini à partir du demi-grand-axe. Alors, le Hamiltonien transformé ne dépend plus que du demi-grand-axe et est donc intégrable. On a cinq intégrales premières indépendantes en involutions alors que trois suffiraient, et cette "sur-intégrabilité" est à l'origine de difficultés spécifiques (voir plus loin).

1.4 Les systèmes quasi-intégrables

Nous avons vu que les flots des systèmes Hamiltoniens intégrables peuvent être étudiés en détail. Ces systèmes présentent un deuxième intérêt qui justifie leur importance : de nombreux problèmes en physique mathématique peuvent être considérés comme des *perturbations* de systèmes intégrables. C'est ce qui motive l'introduction de la classe des systèmes *quasi-intégrables*, que nous allons définir plus précisément dans le cas analytique:

Definition 1.4.1. *Un système Hamiltonien est dit quasi-intégrable s'il existe* $\varepsilon_0 > 0$ *tel que son Hamiltonien* $\mathcal{H}$ *vérifie* $\mathcal{H} \in C^\omega\left(]-\varepsilon_0, \varepsilon_0[\times T^*M, \mathbb{R}\right)$ *et si* $h_0 := \mathcal{H}(0,.) : T^*M \to \mathbb{R}$ *est intégrable.*

En utilisant les variables actions-angles (I,θ) associées à $\mathcal{H}(0,.)$, on se ramène à une famille de Hamiltoniens prenant la forme

$$\mathcal{H}(\varepsilon, I, \theta) = h_0(I) + \varepsilon\mathcal{F}(\varepsilon, I, \theta) \qquad \text{où } \mathcal{F} \in C^\omega\left(\left]-\varepsilon_0, \varepsilon_0\right[\times U \times \mathbb{T}^n, \mathbb{R}\right) \tag{1.4.1}$$

avec U ouvert dans $\mathbb{R}^n$.

En fait pour les problèmes étudiés ici, on pourra considérer sans perte de généralité, la famille de Hamiltoniens

$$\mathcal{H}(\varepsilon, I, \theta) = h_0(I) + \varepsilon f(I, \theta) \qquad \text{où } f \in C^\omega\left(U \times \mathbb{T}^n, \mathbb{R}\right). \tag{1.4.2}$$

Cette situation apparaît notamment pour l'étude du mouvement des planètes dans le système solaire (voir [4]). En effet, si la (faible) interaction mutuelle entre les planètes est négligée, alors le système considéré se découple en plusieurs problèmes de Kepler indépendants et est intégrable. C'est précisément pour l'étude de ce problème que la théorie des perturbations a été initiée au dix-huitième siècle.

Le résultat le plus ambitieux serait de montrer que les systèmes quasi-intégrables sont conjugués à des systèmes intégrables. C'est à dire qu'il faudrait trouver une famille à un paramètre $V_\varepsilon \subset T^*M$ constituée d'ouverts connexes invariants pour le flot hamiltonien associé à $\mathcal{H}(\varepsilon, ., .)$ qui sont canoniquement difféomorphes à $U_\varepsilon \times \mathbb{T}^n$ où U_ε est un ouvert dans $\mathbb{R}^n$. De plus, le hamiltonien $\mathcal{H}(\varepsilon, ., .)$ sur V_ε doit être transformé en $h_\varepsilon(\widetilde{I})$ avec une famille à un paramètre de fonctions analytiques h_ε de $U_\varepsilon \times \mathbb{T}^n$ dans $\mathbb{R}$.

Poincaré (voir [103]) a montré que ce résultat est génériquement faux: l'énoncé et la preuve de ce théorème seront donnés plus loin après l'exposé du principe de moyennisation et des méthodes de moyennisation des perturbations.

1.5 Principe de moyennisation

Pour ce paragraphe et le suivant, on peut consulter le livre d'Arnold, Kozlov et Neishtadt (réf. [4]).

On voit que dans le cas quasi-intégrable, le système (1.1.1) prend la forme :

$$\dot{I} = -\varepsilon \partial_\theta f(I, \theta) \; ; \; \dot{\theta} = \nabla h(I) + \varepsilon \partial_I f(I, \theta)$$

où ∇h est le gradient de h. Les variables sont scindées en deux groupes : celles qui varient sur une échelle temporelle rapide tandis que les autres dérivent lentement, c'est notamment le cas pour les actions.

Le *principe de moyennisation* consiste à remplacer le système initial par sa moyenne temporelle suivant le flot non perturbé Φ_h^t associé à h, c'est à dire que l'on passe à $< H > (I, \theta) = h(I) + \varepsilon < f > (I, \theta)$ avec :

$$< f > (I, \theta) = \lim_{t \to \infty} \left(\frac{1}{t} \int_0^t f\left(I, \theta + s\nabla h(I)\right) ds \right).$$

En fait, cette moyenne va dépendre des relations de commensurabilité qui sont vérifiées par les composantes du vecteur $\nabla h(I)$.

Plus précisément, à un sous-module $\mathcal{M} \subset \mathbb{Z}^n$, on associe une *zone de résonance*

$$\mathcal{Z}_\mathcal{M} = \{ I \in \mathbb{R}^n | k \cdot \nabla h(I) = 0 \iff k \in \mathcal{M} \} \ ,$$

et- pour $\mathcal{M} = \{0\}$ - on note $\mathcal{Z}_0$ la zone non-résonante.

Si le rang de $\mathcal{M}$ est égal à $r \in \{0, \dots, n-1\}$, il existe une transformation symplectique $\phi = \mathcal{R}\theta$ et $I = {}^t\mathcal{R}J$ où $\mathcal{R} \in \mathrm{SL}(n, \mathbb{Z})$ est une matrice unimodulaire dont les r premières lignes constituent une base de $\mathcal{M}$.

Dans les nouvelles variables, le hamiltonien considéré devient $\nabla \widetilde{h}(J) = (0, \omega(J))$ lorsque ${}^t\mathcal{R}J \in \mathcal{Z}_\mathcal{M}$ avec $\omega(J) \in \mathbb{R}^{n-r}$ qui ne vérifie aucune relation de commensurabilité.

Alors le flot linéaire de fréquence $\omega(J)$ sur le tore $\mathbb{T}^{n-r}$ est ergodique et la moyenne temporelle $< \widetilde{f} > (J, \phi)$ tend vers la moyenne spatiale

$$< \widetilde{f} > \left(J_1, J_2, \phi_1\right) = \left(\frac{1}{2\pi}\right)^{n-r} \int \int_{\mathbb{T}^{n-r}} \widetilde{f}\left(J_1, J_2, \phi_1, \phi_2\right) d\phi_2 \qquad (1.5.1)$$

où $(J_1, J_2, \phi_1, \phi_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{T}^r \times \mathbb{T}^{n-r}$.

Ainsi, le hamiltonien moyennisé ne dépend plus des angles rapides et le système considéré ne fait intervenir que des variables qui évoluent lentement (par exemple pour déterminer numériquement les solutions, on peut prendre un pas d'intégration de l'ordre de $1/\varepsilon$).

De manière équivalente, le principe de moyennisation est basé sur l'idée que les termes ignorés dans le champ de vecteur initial entraîne seulement de petites oscillations qui sont superposées aux solutions générales du système moyennisé, notamment on peut énoncer :

Theorem 1.5.1. *Avec les notations précédentes, les actions J_2 conjuguées aux angles rapides ϕ_2 deviennent des intégrales premières du système moyennisé (qui est donc plus simple que le système initial).*

De manière équivalente, on trouve $n - r$ intégrales premières qui sont des combinaisons entières des variables d'action initiales I.

Plus particulièrement, si $\mathcal{M} = \{0\}$ (donc dans la zone $\mathcal{Z}_0$), le système moyennisé (aussi appelé système séculaire dans ce cas) ne dépend plus des angles et est intégrable.

Par contre, l'ensemble des points résonants (i.e. : situé dans une zone $\mathcal{Z}_{\mathcal{M}}$ avec $\mathcal{M} \neq \{0\}$) peut être :

— vide, c'est le cas d'une chaîne d'oscillateurs avec des fréquences $(\omega_1, \dots, \omega_n)$ non résonantes.

— dense, c'est la situation générale lorsque l'application fréquence $\partial_I h(I)$ est localement inversible : $\left|\partial_I^2 h(I)\right| \neq 0$, par exemple c'est le cas pour une particule libre sur le tore $\mathbb{T}^n$ où le Hamiltonien associé est celui d'un rotateur (voir plus haut).

— égal à $\mathbb{R}^n$ tout entier dans le cas où la différentielle du Hamiltonien intégrable n'est pas de rang maximal, on dit qu'il admet une *dégénérescence propre*.

Cette dernière situation apparaît pour le problème de Kepler, où le Hamiltonien considéré ne dépend que du demi-grand-axe de l'ellipse parcourue. Lever cette dégénérescence est une difficulté majeure dans les problèmes de mécanique céleste.

Le principe de moyennisation a été introduit par Lagrange et Laplace dans leurs travaux sur les perturbations séculaires des orbites planétaires (c'est à dire lorsque l'on étudie les solutions du système séculaire intégrable pour le problème des *n* corps avec une faible interaction gravitationnelle autour d'un attracteur central massif). Dans ce cas, le système non perturbé est constitué de *n* problèmes de Kepler découplés donc le Théorème 1.5.1 et la remarque précédente entraînent que les demi-grands-axes des planètes sont des intégrales premières du système moyen si leurs périodes de révolution autour du corps central ne sont pas commensurables. Ce résultat a été démontré par Laplace dans son étude de la stabilité du système solaire (1773).

1.6 Théorie des perturbations classiques

Il s'agit maintenant de vérifier la validité du principe de moyennisation : donc, de s'assurer que les solutions du système complet restent proches de celles du système moyennisé.

Notamment, ceci sera le cas si l'on trouve une transformation $\varepsilon-$proche de l'identité qui conjugue le Hamiltonien initial à sa moyenne suivant les angles rapides ϕ_2 (i.e. : dans les variables adaptées à la résonance étudiée comme dans le paragraphe précédent). On est donc ramené à un problème de forme normale où l'on recherche un système de coordonnées adéquat dans lequel les équations considérées prennent la forme la plus "simple" possible.

Ici, on considèrera une transformation correspondant au flot au temps 1 d'un système gouverné par un Hamiltonien $K(I,\theta) = \varepsilon \widetilde{K}(I,\theta)$, que l'on notera Φ^1_K.

Avec la formule de Taylor et la définition du crochet de Poisson, on obtient

$$F \circ \Phi^1_K = F + \{F, K\} + \int_0^1 (1-u)\{\{F,K\}K\} \circ \Phi^u_K du$$

pour toute fonction vectorielle $F \in C^2(U \times \mathbb{T}^n, \mathbb{R}^m)$.

Ainsi, le Hamiltonien transformé $H \circ \Phi^1_K$ admet le développement suivant en ε

$$H \circ \Phi^1_K = h + \varepsilon \left(f + \{h, \widetilde{K}\} \right) + O\left(\varepsilon^2\right) \ ,$$

donc pour obtenir $H \circ \Phi^1_K = h(I) + \varepsilon < f > + O\left(\varepsilon^2\right)$, on doit résoudre :

$$f + \{h, \widetilde{K}\} = < f > \quad \Longleftrightarrow \quad \nabla h(I) \cdot \partial_\theta \widetilde{K}(I,\theta) = -f + < f > \tag{1.6.1}$$

qui est l'équation de conjugaison linéarisée ou équation *homologique*.

Il s'agit de l'équation centrale de la théorie des perturbations.

On est dans le cadre analytique donc la fonction f admet le développement en série de Fourier $f(I,\theta) = \sum_{k \in \mathbb{Z}^n} f_k(I) \exp(ik\theta)$.

Dans la zone de résonance $\mathcal{Z}_{\mathcal{M}}$, en utilisant l'expression de la moyenne spatiale (1.5.1) et après le changement de variable $(I,\theta) = \left({}^t\mathcal{R}J, \mathcal{R}^{-1}\phi\right)$, on voit que la moyenne temporelle admet le développement $< f > (I,\theta) = \sum_{k \in \mathcal{M}} f_k(I) \exp(ik\theta)$.

Donc la fonction

$$\widetilde{K}(I,\theta) = \sum_{k \notin \mathcal{M}} \frac{f_k(I)}{i\,(k.\nabla h(I))} \exp(ik\theta)$$

fournit une solution formelle de l'équation homologique 1.6.1 définie sur $\mathcal{Z}_{\mathcal{M}}$ puisque les dénominateurs ne s'annulent pas sur cette zone.

On obtient ainsi une transformation qui normalise le Hamiltonien au premier ordre et, par le même procédé, on peut éliminer *formellement* les angles rapides à tous les ordres (i.e. : le même type d'équation homologique apparaît à tout les ordres pour déterminer X_n où $n \geq 1$).

Cette construction s'appelle la *méthode de Lindstedt*.

1.7 Problème des résonances

Si la fréquence $\nabla h(I)$ est résonante, le produit $k \cdot \nabla h(I)$ s'annule pour un certain multi-entier k non nul, et ainsi l'équation homologique pour obtenir une forme normale intégrable (i.e. : correspondant à $\mathcal{M} = \{0\}$) ne possède tout simplement pas de solution formelle. Il n'y a pas de solutions dans ce cas car le flot linéaire de fréquence $\nabla h(I)$ n'est pas ergodique et l'approximation de la perturbation par sa moyenne spatiale complète n'a tout simplement aucun sens.

C'est la situation la plus connue qui correspond, par exemple, au théorème de non-intégrabilité analytique de Poincaré :

Theorem 1.7.1. *On considère un Hamiltonien $\mathcal{H}(\varepsilon, I, \theta) = h_0(I) + \varepsilon f(I, \theta)$ où $\mathcal{H} \in C^\omega(U \times \mathbb{T}^n, \mathbb{R})$ avec U un ouvert dans $\mathbb{R}^n$, qui vérifie les conditions de*

— Non-dégénérescence : l'application fréquence $\partial_I h(I)$ est de rang maximal (i.e. $\left|\partial_I^2 h(I)\right| \neq 0$ sur U).

— Généricité : la perturbation a un développement en série de Fourier

$$f(I, \theta) = \sum_{k \in \mathbb{Z}^n} f_k(I) \exp(ik\theta)$$

complet, i.e. aucun coefficient f_k n'est identiquement nul sur U.

Alors, il n'existe pas de transformation canonique analytique définie sur un ouvert dans U qui transforme $\mathcal{H}(\varepsilon, ., .)$ en un Hamiltonien intégrable.

Proof. comme on l'a vu, cette transformation ne peut être définie que dans la zone de non-résonance ($\mathcal{Z}_0 = \{I \in \mathbb{R}^n$ tels que $k \cdot \nabla h(I) = 0$ si et seulement si $k = 0\}$) qui admet un complémentaire dense avec notre condition de non-dégénérescence. □

Notamment, ceci implique qu'un système Hamiltonien générique *n'est pas intégrable* (voir [88]).

1.8 Problème des petits diviseurs et théorie KAM

Par opposition à la situation précédente, la conjugaison à un système intégrable est formellement possible si la fréquence $\nabla h(I)$ est non résonante, donc sur $\mathcal{Z}_0$ dont le complémentaire est de mesure nulle si la condition de non-dégénérescence du Théorème 1.7.1 est vérifiée. On a alors l'existence d'une solution formelle, mais rien ne garantit la convergence de la solution. En effet, même si $k \cdot \nabla h(I)$ est non nul pour tout $k \in$

$\mathbb{Z}^n \setminus \{0\}$, le produit scalaire peut (et va) devenir arbitrairement petit pour des multi-entiers de longueurs arbitrairement grandes, impliquant la divergence de la série. C'est le fameux phénomène des petits diviseurs.

Poincaré (voir [103]) pensait que la convergence des séries de Lindstedt était "hautement improbable" mais Kolmogorov (réf. [80]) a montré en 1954 (toujours avec la condition de non-dégénérescence du théorème 1.7.1) que la plupart des tores non résonants (i.e. : au-dessus de $\mathcal{Z}_0$) se prolongent en tores invariants sous le flot perturbé lorsque la perturbation est suffisamment petite. Ceci est obtenu en considérant des tores dont les fréquences associées vérifient une condition arithmétique (diophantienne) générique qui permet le controle des petits dénominateurs qui apparaissent dans les calculs. Arnold (voir [2]) a prouvé que le complémentaire de ces tores invariants a une mesure qui tend vers zéro avec la taille de la perturbation et Moser (réf. [91]) a étendu ce résultat aux Hamiltoniens suffisamment différentiables.

Pour un panorama de la théorie de Kolmogorov-Arnold-Moser, on peut se référer aux très bonnes présentations [29], [105] et [54].

On obtient ainsi un résultat de *stabilité en mesure* : la plupart des orbites sont situées sur un tore invariant donc elles sont quasi-périodiques, définies pour tout les temps et perpétuellement stables car les variables d'action varient très peu autour des tores invariants.

Pour $n = 2$, cette propriété de stabilité est même vraie pour toute solution, dans le cadre du théorème KAM iso-énergétique d'Arnold : sur chaque niveau d'énergie, qui est de dimension 3, persiste une famille de tores invariants de dimension 2 telle que chaque composante connexe du complémentaire est bornée. Alors, ou bien la solution est quasi-périodique, ou bien elle est "coincée" entre deux solutions quasi-périodiques, et un argument utilisant la mesure des tores préservés montre que la solution est encore stable avec de faibles variations des variables d'actions.

En 1964, Arnold (réf. [3]) a démontré qu'une telle propriété ne subsiste pas pour $n \geq 3$. Il a construit un exemple de système Hamiltonien à trois degrés de liberté qui possède un grand nombre de tores invariants grâce à la théorie KAM mais qui, conjointement, possède une solution $(\theta(t), I(t))$ telle que

$$\left|I(\tau) - I_0\right| \geq 1$$

avec $\tau = \tau(\varepsilon)$, et ceci pour tout $\varepsilon > 0$; donc, une orbite peut décrire une large partie de l'espace des phases même avec une perturbation arbitrairement petite.

Donc, pour $n \geq 3$, la théorie KAM ne fournit pas de résultat de stabilité valable pour toutes les solutions.

De plus, les tores KAM forment un ensemble de Cantor nulle part dense (donc, d'intérieur vide) et, du point de vue de la physique, il est impossible de déterminer si une condition initiale conduit à une solution quasi-périodique ou pas.

Enfin on peut également mentionner un troisième problème, qui n'est pas lié aux résonances ou petits diviseurs, mais qui est incontournable.

Problème des grands multiplicateurs

Même en l'absence de petits dénominateurs, la méthode de moyennisation conduit en général à des séries divergentes: ceci avait déjà été remarqué par Poincaré, qui écriva :

> *Ce qui empêche la convergence, ce n'est pas la présence de petits diviseurs s'introduisant par l'intégration, mais celle des grands multiplicateurs s'introduisant par la différentiation* (voir [103]).

Il s'agit tout simplement du problème de la convergence du schéma itératif. En admettant que l'on sache faire face aux problèmes des résonances et des petits diviseurs, on peut alors trouver un changement de variables Φ^{K_1} qui élimine la perturbation à l'ordre ε, puis Φ^{K_2} qui élimine la perturbation à l'ordre ε^2 et ainsi de suite, mais il reste à montrer la convergence du produit infini

$$\Phi = \Phi^{K_1} \circ \Phi^{K_2} \circ \ldots \circ \Phi^{K_n} \circ \ldots$$

et c'est une question délicate.

En fait, le problème précédent, dans le cas d'un système analytique, peut se ramener à la convergence d'une série formelle, où le terme général a_n se trouve être de l'ordre de $A^n(n!)^\alpha$, avec $A > 0$, $\alpha > 0$ et qui généralement diverge. Par contre cette série formelle peut être tronquée à un ordre optimal avec un reste qui atteint une taille minimale avant de diverger, c'est ce que l'on appelle une sommation "au plus petit terme". Selon Poincaré, ce sont "des séries convergentes au sens des astronomes mais divergentes au sens des géomètres".

De plus, dans le cas général (i.e. : pas analytique), la croissance de ces "grands multiplicateurs" dans la construction des formes normales dépend uniquement de la régularité du système étudié. On peut toujours faire une sommation "au plus petit terme" pour pallier à la divergence dans le schéma itératif. Ceci se traduit par le fait que l'on fait un nombre fini mais "asymptotiquement infini" d'étapes : pour une perturbation de taille ε fixée, on fait un nombre d'étapes de l'ordre de ε^{-a} où $a > 0$, et donc lorsque ε tend vers zéro, ce nombre tend vers l'infini. Au voisinage d'une zone résonante $\mathcal{Z}_{\mathcal{M}}$ (et particulièrement à l'intérieur du domaine non résonant $\mathcal{Z}_0$), on peut alors normaliser le Hamiltonien initial jusqu'à un reste exponentiellement (resp. polynomialement) petit par rapport à l'inverse de la taille de la perturbation si l'on considère un système analytique (resp. C^k ou Hölder). On en déduit alors un résultat de stabilité exponentielle (resp. polynomial) mais seulement "local" et "partiel" : local dans le sens où il n'est valable que pour les solutions qui restent dans un domaine avec des résonances contrôlées où la forme normale est valide, et partiel puisque le système moyenné n'est plus nécessairement intégrable, et on ne peut contrôler l'évolution des variables d'action que dans certaines directions.

1.9 Théorie de Nekhoroshev

1.9.1 Introduction et énoncé

On vient de voir que la stabilité en temps infini des solutions d'un système Hamiltonien presque-intégrable n'est en général pas vraie, on va donc essayer d'établir la stabilité en temps fini mais très long de ces solutions au sens de la définition suivante :

Definition 1.9.1. *Avec les notations précédentes, on dit qu'un système Hamiltonien presque-intégrable est effectivement stable s'il existe des constantes positives b, c telles que pour toute action initiale $I(0)$:*

$$||I(t) - I(0)|| \leq c\varepsilon^b \quad \textit{où} \qquad |t| \leq T(\varepsilon)$$

avec

$$\lim_{\varepsilon \to 0} \varepsilon\, T(\varepsilon) = +\infty$$

car on veut $T(\varepsilon)$ supérieur au temps de stabilité trivial $1/\varepsilon$.

La propriété précédente entraine que les solutions restent dans un compact et ont donc un temps de vie supérieur à $T(\varepsilon)$. De plus, les variables d'actions deviennent des quasi-intégrales premières sur des temps très longs ce qui permet de localiser les solutions dans l'espace des phases.

Pour $n = 2$, la théorie KAM nous donne des résultats de stabilité perpétuelle, c'est-à-dire $T(\varepsilon) = +\infty$, et on peut montrer que $b = 1/2$ dans ce cas.

Par contre, ceci n'est plus vrai pour $n \geq 3$ grâce à (voir [3]).

On parlera de stabilité *polynomiale* (resp. *exponentielle*) si $T(\varepsilon)$ est d'ordre ε^{-1-a} (resp. $\varepsilon^{-1} \exp(\varepsilon^{-a})$), où $a > 0$.

Pour introduire le problème, on commence par un exemple typique de Hamiltonien qui n'est pas effectivement stable : $h(I_1, I_2) = I_1^2 - I_2^2$. En effet, une dérive des actions $(I_1(t), I_2(t))$ sur un segment de longueur 1 et sur un temps de l'ordre de $1/\varepsilon$ apparaît lorsque l'on ajoute la perturbation $f(\theta_1, \theta_2) = -\sin\left(\theta_1 + \theta_2\right)$ avec la solution du système perturbé $h + \varepsilon f$ donnée par :

$$(I_1(t), I_2(t), \theta_1(t), \theta_2(t)) = (\varepsilon t, \varepsilon t, \varepsilon t^2, -\varepsilon t^2). \tag{1.9.1}$$

Dans cet exemple, l'orbite instable dérive le long de la deuxième diagonale qui correspond à la zone de résonance $\mathcal{Z}_{\mathcal{M}}$ associée au module $\mathcal{M} = \mathbb{Z}(1, -1)$ et le Hamiltonien considéré est déjà sous forme normale (ou moyennisé) dans cette zone de résonance. On note que la vitesse de dérive des actions dans cet exemple est maximale compte tenu de la taille ε de la perturbation, c'est le temps trivial $1/\varepsilon$.

Ce type d'exemple a été introduit initialement par Moser (voir [90]) et la propriété importante est ici le fait que le gradient $\nabla h(I_1, I_2)$ reste orthogonal à la deuxième diagonale ou, de manière équivalente, le gradient de la restriction de h sur la deuxième diagonale est identiquement nulle. Ceci entraine que les solutions du système normalisé avec des conditions initiales dans la zone de résonance $\mathcal{Z}_{\mathcal{M}}$ où $\mathcal{M} = \mathbb{Z}(1,-1)$ ne peuvent pas sortir de $\mathcal{Z}_{\mathcal{M}}$. C'est précisément le point important qui doit être évité pour toutes les zones de résonance afin de garantir une stabilité effective

A partir des considérations de Moser (réf. [90]), dans un article précurseur Glimm (réf. [66]) a été le premier à indiquer des propriétés permettant d'éviter le problème précédent, mais c'est Nekhoroshev (réfs. [94], [95], [96]) dans une série de travaux fondamentaux au début des années 1970 qui a montré que, génériquement au sens topologique et de la mesure, le phénomène précédent ne se produit pas et qu'en général, les systèmes Hamiltoniens presque-intégrables analytiques sont effectivement stables sur des temps exponentiellement longs. L'étude de la théorie de Nekhoroshev est au centre de cette thèse, où l'on clarifie et met en perspective la preuve de cette généricité à la lumière de résultats beaucoup plus récents de géométrie algébrique réelle. Par ailleurs, on étend cette théorie à des systèmes Hamiltoniens de régularité beaucoup plus faible que le cadre analytique initialement considéré.

Nous allons tout d'abord donner un énoncé informel du théorème de Nekhoroshev.

Theorem 1.9.1 (Nekhoroshev, 1977)**.** *On considère un système hamiltonien presque intégrable associé à* $h(I) + f(I,\theta)$ *où* $(I,\theta) \in B_R \times \mathbb{T}^n$*, avec* $B_R \subset \mathbb{R}^n$ *la boule ouverte de rayon* R *centré à l'origine et* $||f|| < \varepsilon$ *où* $||\cdot||$ *est une norme fonctionnelle adaptée suivant la régularité des systèmes étudiés.*

On suppose que :

(i) le système est analytique;

(ii) h satisfait une condition "générique" dite d'escarpement (en anglais, steepness).

Alors il existe des constantes positives C_1*,* C_2*,* C_3*,* a*,* b*,* ε_0 *qui ne dépendent que de* h *telles que, pour* $\varepsilon \leq \varepsilon_0$*, on a :*

$$|I(t) - I(0)| \leq C_1 \varepsilon^b \quad , \qquad |t| \leq \frac{C_2}{\varepsilon} \exp\left(C_3 \varepsilon^{-a}\right)$$

pour toute action initiale $I(0) \in B_{R/2}$.

Les constantes a et b sont appelés exposants de stabilité, la valeur de a est la plus importante car elle fournit le temps de stabilité. La valeur donnée par Nekhoroshev pour l'exposant de stabilité a tend naturellement vers l'infini lorsque n tend vers l'infini et le temps de stabilité tend vers le temps trivial $1/\varepsilon$. Guzzo, Chierchia et Benettin (voir [70]) ont donné une version très raffinée de ce résultat avec des exposants de stabilité qui sont certainement les plus précis que l'on peut obtenir avec la stratégie de Nekhoroshev.

La preuve de ce théorème est expliquée en détail dans la Partie III mais nous en donnons une description informelle ci-dessous.

1.9.2 Aspects analytiques de la preuve

Le résultat de Nekhoroshev peut être étendu aux Hamiltoniens de classe Gevrey (voir [87]) toujours avec un temps de stabilité exponentiel par rapport à l'inverse de la taille de la perturbation, ainsi qu'en régularité C^k ($k \in \mathbb{N}$) (réf. [30]) et Hölder (réf. [14]): dans ces deux derniers cas le temps de stabilité est seulement polynomial par rapport à l'inverse de la taille de la perturbation.

La première étape de la preuve est la normalisation du système presque intégrable jusqu'à un ordre élevé dans une large partie de la zone non-résonante $\mathcal{Z}_0$ ainsi que dans des voisinages des zones résonantes $\mathcal{Z}_{\mathcal{M}}$ pour $\mathcal{M} \neq \{0\}$. Ceci est obtenu par un contrôle des petits dénominateurs avec des conditions arithmétiques similaires à celles qui apparaissent dans la théorie KAM. La taille des restes dans ces formes normales dépend uniquement de la régularité du système étudié et impose le temps de stabilité des solutions.

C'est précisément cette étape qui est considérée dans la Partie III de cette thèse, et qui correspond à l'article [14], où l'on développe une version adaptée à la théorie des perturbations d'un Lemme classique d'approximation analytique des fonctions Hölder dû à Jackson-Moser-Zehnder (voir [45]). Cet outil permet d'étendre à des systèmes de classe Hölder de manière simple et rapide les estimations génériques de Nekhoroshev les plus raffinées obtenues dans le cas analytique par Guzzo, Chierchia et Benettin (voir [70]). On obtient alors des temps de stabilité qui sont polynomiaux par rapport à l'inverse de la taille de la perturbation et donc des bornes à la vitesse de la diffusion d'Arnol'd pour les systèmes de faible régularité. Le schéma de preuve, pour un Hamiltonien presque intégrable H de classe Hölder, consiste à normaliser son lissage analytique H_{An} puis à contrôler précisément la différence $H - H_{An}$ et, enfin, à appliquer les arguments de [70]. C'est le contrôle de l'erreur $H - H_{An}$ qui nécessite des estimées non classiques dans la preuve du Lemme d'approximation analytique pour des fonctions Hölder.

1.9.3 Aspects géométriques de la preuve

Comme on l'a vu, les formes normales que l'on considère au voisinage des résonances non triviales (i.e. engendrées par un module $\mathcal{M} \neq \{0\}$) donnent seulement un contrôle partiel de la dynamique et c'est là que la propriété d'escarpement intervient dans les arguments de Nekhoroshev. En effet, il s'agit d'une condition qui assure l'alternative suivante : soit une solution du système normalisé associée à une condition initiale dans une zone résonante non-triviale varie peu, soit elle s'écarte en un temps rapide de la zone de résonance considérée pour rentrer dans une zone associée à une résonance d'ordre strictement inférieur (i.e. engendrée par un module de résonance de dimension strictement inférieure). Ce schéma dichotomique garantit que pour toute condition initiale appartenant à une zone associée à un module de résonance non-trivial $\mathcal{M} \neq \{0\}$

- soit la solution associée reste bornée à l'intérieur de la zone de résonance initiale pendant un temps très long;

– soit la solution traverse rapidement au plus un nombre fini de zones associées à des résonances non-triviales pour ensuite rentrer dans la zone non-résonante $\mathcal{Z}_0$.

Comme on l'a montré auparavant, toute action en $\mathcal{Z}_0$ a une dérive très lente. De plus, la taille des zones résonantes non-triviales est de l'ordre d'une puissance de la taille de la perturbation ε. Ces arguments permettent donc d'établir la stabilité effective énoncée dans le Théorème de Nekhoroshev.

La définition originale d'escarpement donnée par Nekhoroshev est compliquée et est étudiée de manière très approfondie dans cette thèse. On peut donner une première caractérisation géométrique de cette propriété établie par Ilyashenko (réf. [75]) dans le cas complexe et par Niederman (réf. [98]) dans le cas réel.

Theorem 1.9.2. *Une fonction holomorphe (resp. réelle analytique) est escarpée si et seulement si elle n'a pas de points critiques et si sa restriction à tout sous-espace affine propre n'admet que des points critiques isolés .*

En particulier, cette propriété est vérifiée dans le cas important des fonctions convexes où les points critiques considérés sont non-dégénérés donc isolés. Par contre, la convexité est une propriété ouverte mais pas générique.

Un exemple de fonction non-escarpée est donné par $f(x, y) = x^2 - y^2$ qui correspond au hamiltonien donnant lieu à la solution instable (1.9.1) pour une perturbation arbitrairement petites. Si l'on ajoute un terme d'ordre plus élevé en considérant $g(x, y) = h(x, y) + x^3$, on retrouve une fonction escarpée. Comme on le voit ci dessous, cette dernière propriété avec les fonctions h et g, relève du théorème de généricité qui est au centre de cette thèse.

Plus précisément, Nekhoroshev a démontré, initialement dans [94] puis précisé dans [95] et [96]), l'énoncé suivant :

Theorem 1.9.3. *Pour tout couple d'entiers $n \geq 2$ et $r \geq 2$, le polynôme de Taylor d'ordre r d'une fonction non-escarpée à n variables de classe C^{2r-1} appartient à un sous-ensemble semi-algébrique[1] $\mathcal{NE}$ de l'espace des polynôme à n variables et de degré r.*

De plus, $\mathcal{NE}$ a une codimension qui devient strictement positive pour un degré r suffisament grand (on peut prendre $r \gtrsim \lfloor n/4 \rfloor$). Ce dernier résultat entraine la généricité de la propriété d'escarpement aussi bien au sens topologique qu'au sens de la mesure.

Bien que la théorie de Nekhoroshev soit un sujet d'étude classique en dynamique hamiltonienne, la preuve de la généricité de l'escarpement est restée pratiquement non

[1] C'est à dire, un ensemble dont les points vérifient un nombre fini d'équations et inéquations polynomiales.

étudiée depuis 50 ans ! Ceci est peut être dû au fait qu'elle n'emploie aucun argument concernant les systèmes dynamiques, mais utilise des arguments quantitatifs de géométrie algébrique réelle et d'analyse complexe. En dehors de l'article original ([94]), à notre connaissance les seuls travaux sur le sujet sont [111], [47] et [13] (ce dernier papier constitue la Partie V de cette thèse). Ces travaux utilisent en "boîte noire" le schéma de preuve de Nekhoroshev pour établir des critères explicites qui assurent l'escarpement dans le cas de fonctions à moins de 5 variables avec les coefficients de Taylor de la fonction considérée de degré inférieur ou égal à 5.

Dans la première partie de cette thèse, on revisite la preuve initiale (voir [94]) de la généricité de la propriété d'escarpement puis on démontre des critères explicites généraux qui permettent de déterminer si une fonction donnée suffisamment régulière est escarpée.

On réécrit tout d'abord la preuve de la généricité de l'escarpement à la lumière de résultats beaucoup plus récents. Plus particulièrement, la géométrie algébrique réelle était encore balbutiante au moment où Nekhoroshev a démontré ce résultat et son article (réf. [94]) mélange, avec une rédaction parcellaire et obscure par moment, des preuves de résultats fondamentaux nécessaires, ainsi que des propriétés spécifiques au problème étudié. Tout ceci rend difficile la lecture de ce texte.

Notamment, comme on le verra dans la première moitié de la Partie I, Nekhoroshev prouve, dans le cas particulier qui l'intéresse, un théorème général de Yomdin (réf. [116]) démontré 35 ans plus tard sur la reparamétrisation analytique des ensembles semi-algébriques. Sans trop rentrer dans les détails, cette reparamétrisation d'un ensemble semi-algébrique $A \subset \mathbb{R}^n$ consiste à recouvrir A par une collection finie d'ensemble A_j qui sont chacun l'image du cube unité dans $\mathbb{R}^n$ par une fonction semi-algébrique[2] suffisament régulière ayant des dérivées que l'on peut borner. On a ce contrôle jusqu'à un ordre prescrit dans le cas des reparamétrisations introduites indépendamment par Yomdin [115] et Gromov [67] (ceci s'appelle le Lemme Algébrique de Yomdin-Gromov dans la littérature), et jusqu'à l'infini dans le cas des reparamétrisations analytiques étudiées par Yomdin [116] et qui est considéré ici[3]).

Cette nouvelle paramétrisation contrôlée est centrale dans la preuve de la généricité de la propriété d'escarpement car elle permet de réduire le problème étudié au cas polynomial et la propriété d'escarpement d'une fonction donnée au voisinage d'un point est déterminé par le jet à un ordre fini de cette fonction en ce point. Donc on peut ramener le problème étudié à la dimension finie.

Le second ingrédient central est une analyse fine du système polynomial correspondant au problème étudié après reparamétrisation, ceci permet de borner le rang de ce système et aboutir à la majoration désirée sur la codimension, ce qui entraine la généricité de la propriété d'escarpement.

[2]C'est à dire une fonction dont le graphe est un ensemble semi-algébrique.

[3]Tout ensemble semi-algébrique peut être entièrement reparamétré avec un controle des dérivés jusqu'à un ordre prescrit, alors que dans le cas analytique il faut toujours exclure un nombre fini de petits voisinages de singularités complexes.

Le système polynomial qui apparait dans la preuve précédente permet de trouver des critères explicites pour vérifier la propriété d'escarpement d'une fonction donnée en fonction de ses coefficients de Taylor. C'est ce qui est développé dans la seconde moitié de la Partie I et ces critères sont importants en vu des applications de la théorie de Nekhoroshev. La définition de l'escarpement n'est pas constructive, et il est difficile d'établir si une fonction donnée est escarpée ou non sauf dans le cas particulier des fonctions convexes. Ainsi en mécanique céleste des problèmes importants donnent lieu à des approximations intégrables non convexes, c'est notamment le cas lorsque l'on considère le problème séculaire, ou normalisé, des trois corps dans l'appoximation planétaire (ceci est discuté dans l'introduction de la Partie I, voir aussi [102]). Avant ce travail de thèse, il existait uniquement des critères pour des polynômes de bas degrés et dépendant de peu de variables (réf. [111] puis [13], ce dernier article constitue l'annexe V). On démontre ici des critères généraux valables pour une fonction avec un nombre quelconque de variables et qui portent sur ses coefficients de Taylor à un ordre arbitraire ainsi qu'un nombre fini de paramètres réels externes qui, génériquement, appartiennent à des ensembles compacts.

La Partie II est consacrée à l'extension d'un résultat d'analyse complexe montré par Nekhoroshev dans sa preuve de la généricité de l'escarpement et qui est intéressant en soi. Plus précisément, on montre que :

Theorem 1.9.4. *Soient Ω un domaine borné dans $\mathbb{C}$, $k \geq 1$ un nombre entier et $\mathcal{K} \subset \Omega$ un sous-ensemble compact de cardinalité strictement supérieure à k.*

Alors, pour toute fonction f holomorphe sur $\overline{\Omega}$, dont le graphe est contenu dans la courbe algébrique d'un polynôme de deux variables[4] $S \in \mathbb{C}[Z, W]$ de degré borné par k, la quantité

$$\frac{\max_{\overline{\Omega}} |f|}{\max_{\mathcal{K}} |f|}$$

est bornée par une constante qui ne dépend que de k, Ω, $\mathcal{K}$ mais pas de f.

Remark 1.9.1. Ce type de majoration s'appelle une inégalité de Bernstein-Remez.

Ce résultat a été démontré par Briskin-Yomdin (réf. [38]) et Roytwarf-Yomdin (réf. [107]) dans le cas où $\mathcal{K}$ est un intervalle réel, puis par Yomdin (réf. [117] et Friedland-Yomdin (réf. [63]) dans le cas où $\mathcal{K}$ est un ensemble discret de cardinalité assez élevée, grâce à des arguments de géométrie algébrique réelle et d'analyse complexe. Ici, on généralise les raisonnements utilisés par Nekhoroshev pour prouver une inégalité de Bernstein-Remez dans le cas particulier qui apparaissait pour la preuve de la généricité de l'escarpement. La démonstration développé ici est différente de celle de Yomdin: elle s'appuie sur des théorèmes classiques d'analyse complexe et permet de montrer de manière beaucoup plus simple l'inégalité de Bernstein-Remez que dans [107] et [117] où il y a une preuve constructive, donc très détaillée mais aussi plus délicate.

[4] I.e. f vérifie $S(z, f(z)) = 0$, et on dit que f est une fonction algébrique

La partie IV est encore un travail en cours. On y montre un résultat qui devrait être utile pour prouver une conjecture d'Arnol'd, Kozlov et Neishtadt sur la mesure des tores KAM invariants pour un système presque-intégrable générique, qui devrait avoir une mesure comparable à la taille de la perturbation, alors que les résultats classiques donnent une mesure comparable à la racine carré de la taille de la perturbation (voir [4]).

Biasco et Chierchia (réf. [25]) ont montré que la mesure du complémentaire des tores KAM invariants pour un système mécanique presque-intégrable générique du type $H(I,\theta) = I_1^2/2 + \varepsilon f_1(\theta_1) + \ldots + I_n^2/2 + \varepsilon f_n(\theta_n)$ admet une majoration d'ordre $O(\varepsilon)$: ceci constitue une partie de la conjecture d'Arnol'd-Kozlov-Neishtadt

On prouve ici un résultat qui devrait être utile pour étendre le résultat de Biasco-Chierchia aux hamiltoniens de la forme $H(I,\theta) = I^2/2 + \varepsilon f(I,\theta)$ - et, possiblement, au cas général des hamiltoniens presque-intégrables génériques.

Quand la perturbation f dépend aussi des variables d'actions, les arguments de Biasco-Chierchia ([25]) ne sont plus valides et on propose de surmonter ces obstacles en utilisant la théorie de Morse-Sard quantitative développée par Yomdin (réfs. [114], [119]).

Enfin, la Partie V correspond à l'article [13] dans lequel on donne des conditions explicites suffisantes pour garantir l'escarpement dans le cas d'une fonction avec moins de cinq variables. C'est le cas le plus simple où apparaît la structure des équations qui sont données en toute généralité dans la Partie I. Ce travail est en fait antérieur à la Partie I, mais il permet de voir explicitement sur des exemples les difficultés pour établir les critères de I.

Pour ne pas alourdir la rédaction, les preuves et les énoncés de plusieurs résultats intermédiaires ont été placées dans des appendices à la fin du manuscrit.

Firenze, le 9 Mars 2023

Part I

Semi-algebraic geometry and generic Hamiltonian stability

Abstract

The steepness property is a geometric transversality condition which proves fundamental in order to ensure the stability of nearly-integrable Hamiltonian systems over long timespans. Steep functions were originally introduced by Nekhoroshev, who also proved their genericity: namely, the Taylor polynomials of sufficiently smooth non-steep functions are contained in a semi-algebraic set of positive codimension in the space of polynomials. The demonstration of this result was originally published in 1973 and has been hardly studied ever since, probably due to the fact that it involves no arguments of dynamical systems: it makes use of quantitative reasonings of real-algebraic geometry and complex analysis. The aim of the present work is two-fold. In the first part, the original proof of the genericity of steepness is rewritten by making use of modern tools of real-algebraic geometry: this allows to clarify the original reasonings, that were obscure or sketchy in many parts. In particular, Yomdin's Lemma on the analytic reparametrization of semi-algebraic sets, together with non trivial estimates on the codimension of certain algebraic varieties, turn out to be the fundamental ingredients to prove the genericity of steepness. The second part of this work is devoted to the formulation of explicit algebraic criteria to check steepness of any given sufficiently regular function, which constitutes a very important result for applications. These criteria involve both the derivatives of the studied function up to any given order and external real parameters that, generically, belong to compact sets.

Chapter 2

Introduction

2.1 Hamiltonian formalism and nearly-integrable systems

Hamiltonian formalism is the natural setting appearing in the study of many physical systems. Namely, for any given positive integer n, we consider a symplectic manifold $\mathcal{M}$ of dimension $2n$, endowed with a skew-symmetric two-form ω, and with a function $H \in C^2(\mathcal{M})$ classically called Hamiltonian. A Hamiltonian system on $\mathcal{M}$ is the dynamical system governed by the vector field X_H verifying

$$i_{X_H}\omega := \omega(X_H, \cdot) = dH \ . \tag{2.1.1}$$

In the simplest case, we consider the motion of a point on a n-dimensional Riemannian manifold $\mathcal{R}$ endowed with the euclidean metric - called the configuration manifold - under Newton's second law

$$\ddot{q} = -\nabla U(q) \ ,$$

where U is a smooth potential function, and q is a system of local coordinates for $\mathcal{R}$. This system can be conjugated by duality due to Legendre's transformation and reads

$$\dot{p} = -\partial_q H(p,q) \quad ; \quad \dot{q} = \partial_p H(p,q) \tag{2.1.2}$$

where $H(p,q)$ is a real smooth function on the cotangent bundle $T^\star\mathcal{R}$, and p is the local coordinate conjugated to q. In this example, if one takes $\mathcal{M} \equiv T^\star\mathcal{R}$, and if one chooses (p,q) to be Darboux's coordinates associated to the two-form $\omega(p,q) \equiv \sum_{j=1}^n dp_i \wedge dq_i$, then system (2.1.2) is locally equivalent to (2.1.1).

Among Hamiltonian system, an important rôle is played by those which are integrable by quadrature. Due to the classical Liouville-Arnol'd Theorem, under general topological assumptions, an integrable system depending on $2n$ variables (n degrees of freedom) can be conjugated to a Hamiltonian system on the cotangent bundle of the

n-dimensional torus $\mathbb{T}^n$, whose equations of motion take the form

$$\dot{I} = -\partial_\vartheta h(I) = 0 \quad , \qquad \dot{\vartheta} = \partial_I h(I) \quad ,$$

where $(I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ are called action-angle coordinates. Therefore, the phase space of an integrable system is foliated by invariant tori carrying the linear motions of the angular variables (called quasi-periodic motions).

Integrable systems are exceptional[1], but many important physical problems can be described by Hamiltonian systems which are close to integrable. Namely, the dynamics of a nearly-integrable Hamiltonian system is described by a Hamiltonian function whose form in action-angle coordinates reads

$$H(I, \vartheta) := h(I) + \varepsilon f(I, \vartheta) \quad , \qquad (I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$$

where ε is a small parameter that tunes the size of the perturbation εf w.r.t. the integrable part h.

The structure of the phase space of this kind of systems can be inferred with the help of classical Kolmogorov-Arnol'd-Moser (KAM) theory. Namely, under the generic non-degeneracy condition that ∇h is a local diffeomorphism, a Cantor-like set of positive Lebesgue measure of invariant tori carrying quasi-periodic motions for the integrable flow persists under a suitably small perturbation (see e.g. ref. [5], [46]). As this Cantor-like set is nowhere dense, it is extremely difficult to determine numerically whether a given solution is quasi-periodic or not.

Moreover, for a Hamiltonian system depending on n degrees of freedom (hence a $2n$-dimensional system), the invariant tori provided by classical KAM theory are n-dimensional. Hence, if $n = 2$, any pair of invariant tori disconnects the three-dimensional energy level, so that the solutions of the perturbed system are global and bounded over *infinite* times. However, an arbitrary large drift of the orbits is possible in case $n \geq 3$. Actually, in ref. [3] Arnol'd proposed an example of a nearly-integrable Hamiltonian system where an arbitrary large instability of the action variables occurs for an arbitrary small perturbation. This phenomenon is known under the name of *Arnol'd's diffusion* (see ref. [78] and references therein for the most recent developments in this field). Thus, results of stability for quasi integrable Hamiltonian systems which are valid for an open set of initial condition can only be proved over *finite* times.

2.2 Long time stability of nearly-integrable systems

In the 1970s, Nekhoroshev[2] proved that if we consider a real-analytic, integrable Hamiltonian whose gradient satisfies a suitable, quantitative transversality condition known

[1] Three examples of integrable systems are the classical Kepler's problem, the harmonic oscillator, and Lagrange's top.

[2] See [95], or [70] for a more modern presentation

as *steepness*, then, for any sufficiently small perturbation the solutions of the perturbed system are stable and have a very long time of existence[3].

The original definition of steepness given by Nekhoroshev is quite involved and will be discussed at length in the sequel. In order to grasp an idea of what this property means, it is worth mentioning that a real-analytic function is steep if and only if it has no isolated critical points and if any of its restrictions to any affine proper subspace admits only isolated critical points (see [75] and [98]). This is especially satisfied in the important case of an integrable Hamiltonian which is strictly convex in the action variables, so that all convex functions are *steep*.

Actually, the vast majority of the works on Nekhoroshev's theory concerns small perturbation of convex integrable Hamiltonian but Nekhoroshev also proved in [94] that - unlike convexity - the steepness condition is generic, both in measure and topological sense: the Taylor polynomials of sufficiently high order of non-steep functions are contained in a semi-algebraic set having positive codimension in the space of polynomials. The proof and the refinement of this property constitute the first part of the present work. However, before presenting this and the other main results, we would like to highlight that - even though convex systems are common [4] - non-convex integrable Hamiltonians occur in the investigation of important problems of mechanics.

Namely, we consider a symplectic manifold $(\mathcal{M}, \omega)$ of dimension $2n$, $n \in \mathbb{N}$, where ω is an everywhere non-degenerate closed 2-form, a smooth symplectic vector field X on $\mathcal{M}$ (meaning that the one-form $i_X\Omega$ is closed) and an equilibrium point $p^* \in \mathcal{M}$, that is $X(p^*) = 0$.

We are interested in studying whether p^* is stable or not.

Since we are in a conservative case, a first observation is that, if p^* is stable, then the spectrum of the linearized system around p^* is $\{\pm i\alpha_1, \dots, \pm i\alpha_n\}$ where $\alpha_1, \dots, \alpha_n$ are reals, and p^* is an elliptic equilibrium position.

The problem being local, we can ensure without any loss of generality (this is specified in [32]) that $(M, \Omega) = (\mathbb{R}^{2n}, \Omega_0)$ where Ω_0 is the canonical symplectic structure of $\mathbb{R}^{2n}$, hence $\Omega_0(x, y) = dx \wedge dy$ for the conjuguated variables $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, under generic assumptions (see [32]), we can assume that the considered system derives from a Hamiltonian of the form:

$$H(x, y) = \sum_{j=1}^{n} \alpha_j (x_j^2 + y_j^2)/2 + O_3(x, y), \tag{2.2.1}$$

where our standing assumption from now on is that the Hamiltonian H is real-analytic. Such a system, under a suitable rescaling, can be considered as nearly-integrable.

[3]The time of stability depends of the regularity of the considered system and is exponential (polynomial) in the inverse of the size of the perturbation if the total Hamiltonian belongs to the Gevrey (Hölder) class. See [87], [30], [14].

[4]See e.g. [97], [15] in the study of the three-body problem, [7], [8] in the context of central force motions, and [9], [6], [106], [56] in the framework of infinite-dimensional Hamiltonian systems.

In this setting, there are two cases for which one knows that stability holds true for the considered equilibrium.

The first case is when the quadratic part H_2 is sign-definite, or, equivalently, when the components of the vector $\alpha \in \mathbb{R}^n$ have the same sign. Indeed, the Hamiltonian function has then a strict minimum (or maximum) at the origin, and as this function is constant along the flow (it is in particular a Lyapounov function) one can construct, using standard arguments, a basis of neighborhoods of the origin which are invariant, and the latter property is obviously equivalent to stability.

The second case is when $n = 2$ and when the so called Arnol'd's iso-energetic non-degeneracy condition is satisfied. Then, KAM stability occurs in every energy level passing sufficiently close to the origin, implying Lyapounov stability, due to the fact that the two-dimensional tori disconnect each three-dimensional energy level (see for instance [1] and [91]). It is easy to see that the Arnold iso-energetic non-degeneracy condition is generic in measure and topology as a function of the coefficients of the $O_4(x, y)$ part of the Taylor expansion of H around the origin.

In the other cases, a large unstability due to Arnold diffusion can occur (see [57]), but it has been proved in [32] that, generically, any solution starting sufficiently close to the equilibrium point remains close to it for an interval of time which is double-exponentially large ($\exp \circ \exp$) with respect to the inverse of the distance to the equilibrium point. The latter result is obtained by making use of Nekhoroshev's theory and relies crucially on the genericity of steep functions, since one needs to build a suitable steep integrable approximation of the complete system.

The same issue arises in order to apply Nekhoroshev's theory to concrete examples. Especially, in Celestial Mechanics, we have important problems where an elliptic equilibrium arises with a quadratic term in (2.2.1) which is not sign definite: this is the case for the Lagrange's equilibrium points L4, L5 in the restricted three body problem (see [19]) and in the averaged ("secular") planetary three body problem (this is due to the Herman's resonance, see [60] and [86]). The latter system is a crucial approximation to apply Hamiltonian perturbation theory (hence KAM or Nekhoroshev theory) in Celestial Mechanics. Moreover, we cannot always build an integrable approximation of this kind of systems which is convex in action variables, hence we have to consider steep non convex Hamiltonians in order to infer stability results with the help of Nekhoroshev's theory. For the study of the Lagrange's equilibrium points, it is possible in most cases to recover steepness by considering higher order approximation (see [19]), actually this corresponds to general considerations on functions with three variables which will be specified in the sequel. For the secular planetary three-body problem, the associated Hamiltonian is not convex w.r.t. the actions (see [102]) and much more variables are involved than for the Lagrange's points, hence we really need new criteria to ensure that a given function is steep or not in this case. Up to now, generic explicit conditions for steepness were known only for functions of three (the conditions given by Nekhoroshev in [95]), four (see [111]) or five variables (see [13]). The second part of this work is devoted to proving explicit conditions for steepness which are generic for functions of an arbitrary number of variables.

It can also be specified that steepness is a necessary condition in order to ensure long-time stability: if the steepness condition is dropped, large instablities may occur over times of order $1/\varepsilon$, which is the shortest possible time of drift when considering perturbations of magnitude $O(\varepsilon)$ (see [98] and [34]).

In the context of KAM theory, Herman (see [73]) has shown that the lack of steepness of the integrable Hamiltonian allows to build perturbation for which one can find a $G_\delta-$dense set of initial conditions leading to orbits whose action components are unbounded while the integrable Hamiltonian can also be Kolmogorov non-degenerate, hence most of the orbits lie on invariant tori and we have simultanously, existence of large zones of stability and unstability.

Steepness also arises in the framework of Arnol'd's diffusion (see [21]) for the optimality of the time of diffusion. Finally, recent works of Bambusi and Langella [11] show that Nekhoroshev's classical proof of stability for perturbations of steep integrable Hamiltonian systems is also relevant in the study of PDE's, considered as infinite-dimensional Hamiltonian systems.

2.3 Genericity and explicit criteria for steepness

Now, we specify Nekhoroshev's effective result of stability (see refs. [95], [96]), which is valid for an open set of initial conditions provided that the total Hamiltonian is regular enough and that its integrable part satisfies the following transversality property on its gradient:

Definition 2.3.1 (Steepness)**.** Fix $\delta > 0$, $R > 0$. A C^2 function $h : B^n(0, R+2\delta) \to \mathbb{R}$ is *steep* in $B^n(0, R)$ with *steepness indices* $\alpha_1, \dots, \alpha_{n-1} \geq 1$ and *steepness coefficients* $C_1, \dots, C_{n-1}, \delta$ if:

1. $\inf_{I\in B^n(0,R)} ||\nabla h(I)|| > 0$;

2. for any $I \in B^n(0, R)$, for any integer $1 \leq m < n$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla h(I)$ and endowed with the induced euclidean metric, one has:

$$\max_{0\leq\eta\leq\xi} \min_{u\in\Gamma^m,\,||u||_2=\eta} ||\, \pi_{\Gamma^m} \nabla h(I+u)\, ||_2 > C_m \xi^{\alpha_m}, \quad \forall \xi \in (0,\delta], \tag{2.3.1}$$

   where π_{Γ^m} stands for the orthogonal projection on Γ^m.

Remark 2.3.1. Since in definition 11.2.1 the subspace $\Gamma^m \subset \mathbb{R}^n$ is endowed with the induced metric, for all $u \in \Gamma^m$ one has $||\pi_{\Gamma^m} \nabla h(I+u)||_2 = ||\nabla(h|_{I+\Gamma^m})(I+u)||_2$, where $h|_{I+\Gamma^m}$ indicates the restriction of h to the affine subspace $I + \Gamma^m$.

As it is showed in [98], in the analytic case a function is steep if and only if, on any affine hyperplane $I + \Gamma_m$, there exists no curve γ with one endpoint in I such that the restriction $\nabla(h|_{I+\Gamma^m})$ vanishes identically on γ. From a heuristic point of view, for any value $m \in \{1, ..., n-1\}$ the gradient ∇h must "bend" towards $I + \Gamma^m$ when "travelling" along the curve $\gamma \in I + \Gamma^m$, so that critical points for the restriction of h to $I + \Gamma^m$ must not accumulate.

With such a notion, Nekhoroshev's effective result of stability reads

Theorem 2.3.1 (Nekhoroshev, 1977)**.** *Consider a nearly-integrable system governed by Hamiltonian*

$$H(I, \vartheta) := h(I) + \varepsilon f(I, \vartheta) \quad , \qquad H \in C^{\omega}(B_r \times \mathbb{T}^n) \, ,$$

where $B^n(0, r)$ is the open ball of radius r in $\mathbb{R}^n$, and h is assumed to be steep. Then there exist positive constants $a, b, \varepsilon_0, C_1, C_2$ such that, for any $\varepsilon \in [0, \varepsilon_0)$ and for any initial condition not too close to the boundary, one has $|I(t) - I(0)| \leq C_2 \varepsilon^a$ for any time t satisfying $|t| \leq C_1 \exp\left(\varepsilon^{-b}\right)$.

Remark 2.3.2. The presence of a bound of the kind $|I(t) - I(0)| \leq C_2 \varepsilon^a$ on the variation of the action variables is a consequence of the steepness property. The time of stability depends on the regularity of the function H at hand. In the original formulation by Nekhoroshev, H was considered to be real-analytic, which yielded an exponentially-long time in the inverse of the size of the pertubation (see also [70]). Exponentially-long times of stability hold also in case H is Gevrey (see [87]), whereas only polynomially-long times of stability can be ensured for C^∞ and Hölder functions (see refs. [10], [14]).

As it has already been anticipated in the previous paragraph, the steepness property is generic - both in measure and in topological sense - in the space of Taylor polynomials of sufficiently high order of smooth functions. Namely, let $r, n \geq 2$ be two positive integers. We indicate by $\mathcal{P}(r, n) \subset \mathbb{R}[x_1, \ldots, x_n]$ the subspace of real polynomials in n variables having degree bounded by r. For any point $I_0 \in \mathbb{R}^n$, and any function f of class C^r near I_0, we call *r-jet of f at I_0* the Taylor polynomial of f up to order r calculated at I_0. Moreover, we say that a subset $\mathcal{A} \subset \mathbb{R}^n$ is *semi-algebraic* if it is the finite union of subsets determined by a finite number of polynomial equalities or inequalities (see Definition A.1.1). Nekhoroshev proved in [94]- [96] that

Theorem 2.3.2 (Nekhoroshev, 1973-1979)**.** *The r-jets of all non-steep functions of class C^{2r-1} around a non-critical point $I_0 \in \mathbb{R}^n$ are contained in a semi-algebraic subset $\Omega(r, n)$ of $\mathcal{P}(r, n)$. Moreover, the codimension of $\Omega(r, n)$ in $\mathcal{P}(r, n)$ becomes positive for $r \gtrsim [n^2/4]$.*

Although Nekhoroshev's Theory has been a classic subject of study in the dynamical systems community for more than forty years, the proof of Theorem 2.3.2 has remained poorly understood. This is possibly due to the fact that such a demonstration does not involve any arguments of dynamical systems, but combines quantitative reasonings of real-algebraic geometry and complex analysis. Moreover, real-algebraic geometry was at a more rudimentary level than nowadays at the time that Nekhoroshev's was writing; for this reason, important properties of real-algebraic geometry are discovered[5] in [94] at the same time that they are used to prove Theorem 2.3.2, which makes that work obscure in many parts. In addition, the proofs of some lemmas in that work are sketchy or presented in an old-fashioned way. For these reasons, the first part of this work is devoted to proving and refining Theorem 2.3.2 by making use of modern results of real-algebraic geometry. As we will discuss in detail in the sequel, Yomdin's Lemma about the analytic reparametrization of semi-algebraic sets (see [116]) turns out to be the fundamental ingredient of real-algebraic geometry which is used in the proof of the genericity of steepness.

Moreover, since the definition of steepness is not constructive, it is difficult to directly establish whether a given function is steep or not. Up to the author's knowledge, there are only three articles on this topic (see [111], [47], [13]) which concern only polynomials of low degree depending on a small number of variables. Actually, by developing the arguments used by Nekhoroshev to prove Theorem 2.3.2, it is possible to deduce explicit sufficient algebraic conditions for steepness involving the derivatives up to an arbitrary order of functions depending of an arbitrary number of variables. This proves fundamental for applications of Nekhoroshev's theory to physical models. The second part of this work is dedicated to this topic. Namely, we will prove refined versions of the Theorems below.

Theorem 2.3.3. *The semi-algebraic set $\Omega(r,n)$ in Theorem 2.3.2 satisfies*

$$\Omega(r,n) = closure\left(\bigcup_{m=1}^{n-1} Proj_{\mathcal{P}(r,n)} Z(r,m,n)\right) \tag{2.3.2}$$

where $Z(r,m,n)$ is a semi-algebraic set of $\mathcal{P}(r,n)\times\mathbb{R}^K\times\mathbb{R}^n\times\mathtt{U}(m-1,n)$, $K = K(r,m)$ is a suitable positive integer, and $\mathtt{U}(m-1,n)$ is the compact $m-1$-dimensional Stiefel manifold in $\mathbb{R}^n$ (see section 3 for its definition).

Moreover, for any $m \in \{1,\dots,n-1\}$, the form of $Z(r,m,n)$ can be explicitly computed.

Remark 2.3.3. Theorem 2.3.3 is a first example of an explicit criterion for steepness. Infact, as it is known, there exist explicit general algorithms of real-algebraic geometry that allow to compute the explicit form of the projection and the closure of any given semi-algebraic set (see e.g. [18]). Hence, at least in principle, it would be possible to compute the r.h.s. of (2.3.2) - hence $\Omega(r,n)$ - as the form of $Z(r,m,n)$ is known due to Theorem 2.3.3. However, the complexity of the classic algorithms grows double-

[5]For example, it is remarkable that, up the author's knowledge, a fundamental Bernstein's inequality for algebraic functions is proved for the first-time in Nekhoroshev's work (see [16]). Such a result seems to have passed unnoticed, until it has been widely rediscovered and generalized in the late nineties by Roytwarf and Yomdin in [107], and subsequently developed by several authors.

exponentially in the number of variables, so that they are of little use in practice (see [72]).

Remark 2.3.4. Alternatively, one could use Theorem 2.3.3 in order to check if a function h of class C^{2r-1} around a point I_0 is steep in the following way. Indicating by $\mathtt{T}_{I_0}(h,r,n)$ the r-jet of h at I_0, by (2.3.2) one could check whether there exists $\tau > 0$ such that, for any $m \in \{1,\dots,n-1\}$, and for any choice of parameters $\beta \in \mathbb{R}^K \times \mathbb{R}^n \times \mathtt{U}(m-1,n)$, the pair $(\mathtt{T}_{I_0}(h,r,n),\beta)$ lies outside of $Z(r,m,n)$. This would guarantee that $\mathtt{T}_{I_0}(h,r,n)$ lies outside of closure $\left(\bigcup_{m=1}^{n-1} \mathrm{Proj}_{\mathcal{P}(r,n)} Z(r,m,n)\right)$, so that (2.3.2) and Theorem 2.3.2 would ensure steepness. This is indeed one possibility, and we will make it more explicit in the next section (see Theorem B). However, this criterion involves checking an explicit condition for a non-compact set of parameters (the first components of the vectors β above lie in $\mathbb{R}^K \times \mathbb{R}^n$, whereas the remaining ones belong to the compact Stiefel manifold $\mathtt{U}(m-1,n)$). As we show below, on "most subspaces" steepness can be checked by making use of criteria involving only parameters belonging to a compact set.

Namely, let h be a function of class C^{2r-1} around the origin, satisfying $\nabla h(0) \neq 0$. Then,

Theorem 2.3.4. *It is possible to find explicit algebraic criteria involving the derivatives of h up to order r that ensure that h is steep on the one-dimensional subspaces around the origin.*

Moreover, for any $m \in \{2,\dots,n-1\}$, one has the following properties.

1. *h is steep at the origin on the m-dimensional subspaces orthogonal to $\nabla h(0) \neq 0$ on which the restriction of the hessian $D^2 h(0)$ is non-degenerate.*

2. *On the m-dimensional subspaces of $\nabla h(0)^\perp$ on which the restriction of $D^2 h(0)$ has exactly one null eigenvalue, it is possible to construct explicit algebraic criteria for steepness that involve the r-jet of h at the origin and a finite number of real parameters lying in a compact subset. These criteria can be constructed starting from the explicit form of subset $Z(r,m,n)$ in Theorem 2.3.3 by the means of algorithms involving only linear operations*[6].

Therefore, explicit criteria for steepness involving only the r-jet of h exist in case $m = 1$. In case $m \in \{2,\dots,n-1\}$, instead, with the exception of the m-dimensional subspaces of $\nabla h^\perp(0)$ on which the restriction of $D^2 h(0)$ has two or more null eigenvalues, steepness can be checked by using a criterion which is simpler than those stated in Remarks 2.3.3-2.3.4. Moreover, for any value of $m \in \{1,\dots,n-1\}$, the Hessian of a generic function h is non-degenerate on most subspaces of the m-dimensional Grassmannian $\mathtt{G}(m,n)$, as the following result shows.

[6]Hence, much simpler algorithms than the general algorithms of real-algebraic geometry.

Theorem 2.3.5. *Consider an integer $m \in \{2, \dots, n-1\}$. For any bilinear, symmetric, non-degenerate form $\mathtt{B} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, the m-dimensional subspaces on which the restriction of $\mathtt{B}$ is degenerate are contained in a submanifold of codimension one in the Grassmannian $\mathtt{G}(m, n)$.*

2.4 Informal presentation of the proofs

Roughly speaking, the proof of Theorem 2.3.2 is split into two parts:

– General considerations of semi-algebraic geometry allow to prove that the complicated condition 11.2.1 arising in the definition of steepness is an open property in the space of polynomials $\mathcal{P}(r, n)$. Namely, if (11.2.1) holds for a given polynomial $Q \in \mathcal{P}(r, n)$, then it holds also in a neighborhood of Q with the uniform indices $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m$ and uniform coefficients $C_1, \dots, C_m, \delta$.

– Also, condition (11.2.1) is not satisfied if and only if the Taylor polynomial of h satisfies a certain number of algebraic equations. A detailed analysis of these equations ensures that they only admit a non generic set of solutions.

In the present work, we have results on the two sides of the proof.

2.4.1 Reparametrization of semi-algebraic sets and Bernstein's inequality

We revisit Nekhoroshev's reasonings of semi-algebraic geometry under the light of more recent results in the field.

Due to (11.2.1), a central point to check steepness of a given function h at a point $I \in \mathbb{R}^n$ consists in ensuring a minimal growth of the projection of its gradient on any affine subspace orthogonal to $\nabla h(I) \neq 0$. For a fixed affine subspace $I + \Gamma$ equipped with local coordinates and with the induced euclidean metric, by Remark 11.2.1 the projection of $\nabla h(I)$ on Γ corresponds to the gradient of the restriction $h|_{I+\Gamma}$ expressed in the local coordinates. Hence, one is led to study the locus of minima of $||\nabla h|_{I+\Gamma}||$. By the above considerations, without entering into too many technicalities, a crucial step in Nekhoroshev's proof of the genericity of steepness consists in considering, for any fixed polynomial $P \in \mathbb{R}[X_1, \dots, X_m]$, the semi-algebraic set - called thalweg nowadays (see [28]) - determined by :

$$\mathbb{R}^m \supset \mathcal{T}_P := \left\{u \in \mathbb{R}^m |\ ||\nabla P(u)|| \leq ||\nabla P(v)||\ \forall v \in \mathbb{R}^m \text{ s.t. } ||u|| = ||v||\right\}. \quad (2.4.1)$$

Nekhoroshev shows that $\mathcal{T}_P$ contains the image of a semi-algebraic curve[7] γ which admits a holomorphic extension with the exception of a finite set of singular complex

[7] I.e. a curve having semi-algebraic graph, see also Definition A.1.3.

points whose cardinality depends only on the degree of P and on the number of variables. In particular, one can ensure the existence of a uniform real interval of analyticity and of a uniform complex analyticity width for the curve γ, independently on the choice of the polynomial $P \in \mathcal{P}(r, m)$. More specifically, the graph of γ can be parametrized by analytic-algebraic[8] maps, and the existence of a Bernstein's-like inequality controlling uniformly the growth of this kind of functions in the complex plane ensures uniform upper bounds on the derivatives of these charts (see [107], [116], [117], [16] and references therein for a modern presentation).

Actually, this result about the thalweg in [94] is a particular case of a general theorem due to Yomdin [116] about analytic reparametrizations of semi-algebraic sets (the finitely-differentiable case was firstly stated by Yomdin and Gromov in refs. [115], [67] and then proved by Burguet in [44]). Generally speaking, the reparametrization of a semi-algebraic set A is a subdivision of A into semi-algebraic pieces A_j each of which is the image of a semi-algebraic function[9] of the unit cube. The uniform control on the parametrization of the curve γ is unavoidable in [94], since it ensures that - for a smooth function - steepness is an open property.

Moreover, it is proved that the coefficients of the Taylor expansions of non-steep functions satisfy suitable algebraic conditions (one has a "finite-jet" determinacy of steepness). In this way, the study of the genericity of steepness is reduced to a finite-dimensional setting which involves polynomials of bounded order and this is another crucial step in order to prove the genericity.

It is worth adding some remarks about the fact that Nekhoroshev proves a kind of Bernstein's inequality for algebraic functions (see [94], Lemma 5.1, p.446). Namely, Nekhoroshev proves that an algebraic function which is real-analytic over a real interval I admits a bound on its growth on the complex plane which only depends on its maximum over I and on a constant depending on the degree of the polynomial solving its graph and on the size of its complex domain of holomorphy. This result is proved by exploiting the properties of algebraic curves of complex polynomials in two variables, and by making an intensive use of complex analysis. The original statements are difficult to disentangle from the context of the genericity of steepness and the proofs are very sketchy. This is different from the strategy used by Roytwarf and Yomdin (see [107], [117] and references therein) which relies on arguments of analytic geometry. Since we have not been able to find any reference that shows Nekhoroshev's proof of Bernstein's inequality in detail except for the original paper [94], we have clarified and extended Nekhoroshev's reasonings in [16], and this allowed to obtain a simpler proof of recent results of complex analysis.

It is also worth mentioning that, in connection with arithmetic, the steepness condition is introduced to prevent the abundance of rational vectors on certain sets and it can be noticed that deep applications of the controlled analytic reparametrization of

[8] I.e. analytic maps whose graph solves a non-zero polynomial of two variables.

[9] That is, a function whose graph is a semi-algebraic set, see also Definition A.1.3.

semi-algebraic sets yield bounds on the number of integer points in semi-algebraic sets (see [26] and [50]). In the future, these ideas may help to spread light on the connection between the stability of nearly-integrable Hamiltonian systems and the arithmetic properties of semi-algebraic sets.

We also mention that, in the study of PDEs, the Yomdin-Gromov's algebraic lemma was used by Bourgain, Goldstein, and Schlag [37] to bound the number of integer points in a two-dimensional semi-algebraic set.

2.4.2 Degeneracy condition

We describe heuristically the second part of the proof of Theorem 2.3.2. To make things simple, we restrict this informal discussion to the case of a real-analytic function h around the origin. By formula (11.2.1), if h is non-steep at the origin, then for some $m \in \{1, \dots, n-1\}$ there exists a m-dimensional subspace Γ^m and a curve $\gamma \subset \Gamma^m$ starting at the origin on which the projection $(\pi_{\Gamma^m} \nabla h)|_\gamma$ is identically null. Assuming that γ is sufficiently regular, this means that $(\pi_{\Gamma^m} \nabla h)|_\gamma$ has a zero of infinite order at the origin. It can also be shown (see Theorem 5.0.1) that the the curve γ on which such a condition is satisfied must possess a precise form. By these arguments, one can write down explicitly the equations imposing to the derivatives of the function $(\pi_{\Gamma^m} \nabla h)|_\gamma$ to be all identically null. Moreover, by complicated computations it is possible to check that these equations are all linearly independent. Then, estimates on the codimension of a projected set show that, if one bounds the order of the derivatives that are being considered in the equations by $r \in \mathbb{N}, r \geq 2$, the Taylor polynomials of non-steep functions belong to a semi-algebraic set of positive codimension in the space of polynomials $\mathcal{P}(r, n)$. It is this kind of computations - which are expressed explicitly for the first time in this work - that allow to prove Theorem 2.3.3.

Moreover, by construction, the equations that we are considering depend also on the Taylor coefficients of the curve γ and on the vectors spanning the considered subspace Γ^m. This explains the presence of the space of parameters $\mathbb{R}^K$ and of the Stiefel manifold in the statement of Theorem 2.3.3. On the one hand, by suitably exploiting the form of the equations, one can prove Theorem 2.3.4. On the other hand, Theorem 2.3.5 is independent and its proof relies on the construction of a suitable system of coordinates for the Grassmannian.

2.4.3 Structure of the work

This work is organized as follows: section 3 sets the main notations and definitions, whereas the main results (refined versions of Theorems 2.3.2-2.3.3-2.3.4-2.3.5) are stated in section 4. Section 5 construction of the thalweg and the reparametrization of semi-algebraic sets, whereas section 6 is devoted to the study of the degeneracy condition described in the paragraph above. Section 7 puts together the results of sections 5 and 6 in order to prove the genericity of steepness. Finally, sections 8-9-10 contain the proof of the explicit criteria for steepness.

Chapter 3

Main notations and definitions

Norms

For any $n \in \mathbb{N}^\star$, and for any $\nu \in \mathbb{N}^\star \cup \{\infty\}$, we denote by $||\cdot||_\nu$ the standard ℓ^ν-norm in $\mathbb{R}^n$.

For any integer $q \geq 1$, and for any open subset D of $\mathbb{R}^n$, the symbol $C^q(D)$ indicates the set of q-times continuously differentiable maps $f : D \to \mathbb{R}$. Moreover, we indicate by $C^q_b(D)$ the subset of $C^q(D)$ containing those functions f satisfying

$$\|f\|_{C^q(D)} := \sup_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq q}} \sup_{x\in D} |\partial^\alpha f(x)| < +\infty \,. \tag{3.0.1}$$

In particular, $\left(C^q_b(D), \|\cdot\|_{C^q(D)}\right)$ is a Banach space with multiplicative norm[1].

Sets

In the sequel, we will make use of the following notations:

- For any $d > 0$ and for any $f \in C^q_b(D)$, the symbol $\mathfrak{B}^q(f, d, D)$ indicates the infinite-dimensional open ball of radius d centered at f for the norm (3.0.1);

- $\mathcal{D}_\rho(z_0)$ indicates the open complex disc of radius $\rho > 0$ centered at $z_0 \in \mathbb{C}$;

- $B^n(I, R)$ indicates the real ball of radius $R > 0$ centered at $I \in \mathbb{R}^n$.

- For any connected set $\mathcal{A} \subset \mathbb{C}$, we denote the complex polydisk of width r around $\mathcal{A}$ by

$$(\mathcal{A})_r := \left\{ z \in \mathbb{C} \text{ s.t. } ||z - \mathcal{A}||_2 := \inf_{z'\in\mathcal{A}} ||z - z'|| < r \right\} \,.$$

[1] That is, satisfying an inequality of the form $|fg| \leq K|f|\,|g|$ for a suitable constant K.

Notations of real-algebraic geometry

For any pair (r, n) of positive integers, and for any function h of class C^r in a neighborhood of some point $I_0 \in \mathbb{R}^n$, we denote by

- $\mathcal{P}(r, n) \subset \mathbb{R}[X_1, \dots, X_n]$ the subspace of polynomials over the real field in n real variables with zero constant term and whose degree is bounded by r;
- $\mathcal{P}^\star(r, n) \subset \mathcal{P}(r, n)$ is the subset of those polynomials Q that verify $\nabla Q(0) \neq 0$;
- $\mathtt{T}_{I_0}(h, r, n) \in \mathcal{P}(r, n)$ the Taylor polynomial at order r of the function $h(I) - h(I_0)$ centered at I_0.

Now, let k, n be positive integers, with $k \leq n$.

- We indicate by $\mathtt{G}(k, n)$ the k-dimensional Grassmanniann in $\mathbb{R}^n$, i.e. the compact manifold of k-dimensional linear subspaces in $\mathbb{R}^n$.
- We also denote by $\mathtt{U}(k, n)$ the compact k-dimensional Stiefel manifold in $\mathbb{R}^n$, that is the manifold of ordered orthonormal k-tuples of vectors in $\mathbb{R}^n$.

For any set $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^m$, we indicate by $\Pi_x \mathcal{A}$ its projection onto the first n coordinates, that is the set

$$\Pi_x \mathcal{A} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m | (x, y) \in \mathcal{A}\} .$$

Finally, let m, n be positive integers satisfying $m \leq n$, and let $\{u_1 \dots, u_m\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. For each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, we denote the i-th component of the vector u_j by u_j^i. For any multi-index $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{N}^m$, we set $|\mu| := ||\mu||_1$. Given $I_0 \in \mathbb{R}^n$ and a function h of class $C^{|\mu|}$ in a neighborhood of I_0, we also denote by

$$h_{I_0}^{|\mu|}[\overbrace{u_1}^{\mu_1}, \dots, \overbrace{u_m}^{\mu_m}] := \sum_{\substack{i_{11}, \dots, i_{1\mu_1} = 1 \\ \dots \\ i_{m1}, \dots, i_{m\mu_m} = 1}}^{n} \frac{\partial^{|\mu|} h(I_0)}{\partial I_{i_{11}} \dots \partial I_{i_{1\mu_1}} \dots \partial I_{i_{m1}} \dots \partial I_{i_{m\mu_m}}} u_1^{i_{11}} \dots u_1^{i_{1\mu_1}} \dots u_m^{i_{m1}} \dots u_m^{i_{m\mu_m}} \tag{3.0.2}$$

the μ-th order multilinear form associated to the μ-th coefficient of the Taylor expansion around I_0 of the restriction of h to Span $(u_1, \dots, u_m)$.

Chapter 4

Main results

4.1 Genericity of steepness

As we discussed in Theorem 2.3.2 in the Introduction, the steepness property is generic, both in measure and topological sense, in the space of jets of functions of sufficiently high regularity. In this paragraph, we will give a more quantitative version of this result. Namely, the statement below is a refined version of Nekhoroshev's Theorem on the genericity of steepness, which can be found in refs. [94]- [96].

Theorem (A). *Let $r, n \geq 2$ be two integers, and let $\mathtt{s} := (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$ be a vector satisfying $1 \leq s_m \leq r-1$ for all $m = 1, \dots, n-1$.*

There exists a closed semi-algebraic subset $\boldsymbol{\Omega}_n^{r,\mathtt{s}}$ of $\mathcal{P}(r,n)$ such that, for any $I_0 \in \mathbb{R}^n$, for any real number $\varrho > 0$, for any open, bounded domain $\mathscr{D} \subset C_b^{2r-1}(\overline{B}^n(I_0, \varrho))$, and for any function h satisfying

1. $h \in \mathscr{D}$,
2. $\nabla h(I_0) \neq 0$,
3. $\left|\left|\mathtt{T}_{I_0}(h,r,n) - \boldsymbol{\Omega}_n^{r,\mathtt{s}}\right|\right|_\infty := \inf_{Q \in \boldsymbol{\Omega}_n^{r,\mathtt{s}}} ||\mathtt{T}_{I_0}(h,r,n) - Q||_\infty > \tau > 0$,

one can introduce positive constants $\overline{\epsilon} = \overline{\epsilon}(r, \mathtt{s}, \tau, n)$, $R = R(r, \mathtt{s}, \tau, n, \varrho)$, $C_m = C_m(r, s_m, \tau, n)$ - where $m = 1, \dots, n-1$ - and $\delta = \delta(r, \mathtt{s}, \tau, n, \varrho, \mathscr{D})$ so that any function

$$f \in \mathscr{D} \quad , \qquad f \in \mathfrak{B}^{2r-2}(h, \varepsilon, \overline{B}^n(I_0, \varrho)) \qquad \varepsilon \in [0, \overline{\epsilon}]$$

is steep in $B^n(I_0, R)$, with steepness coefficients C_m, δ and with steepness indices bounded by

$$\overline{\boldsymbol{\alpha}}_m(s) := \begin{cases} s_m \quad , & \text{if } m = 1 \\ 2s_m - 1 \quad , & \text{if } m \geq 2 \ . \end{cases} \tag{4.1.1}$$

Moreover, in $\mathcal{P}(r,n)$ one has

$$\textit{codim}\ \boldsymbol{\Omega}_n^{r,\mathtt{s}} \geq \max\left\{0, \min_{m\in\{1,\dots,n-1\}}\{s_m - m(n-m-1)\}\right\} . \tag{4.1.2}$$

With the setting of Theorem (4.1), we give the following

Definition 4.1.1. A function $f \in C_b^{2r-1}(\overline{B}^n(I_0,\varrho))$ satisfying the hypotheses of Theorem A is said to be steep at order r at the point I_0 for the vector $\mathtt{s}$. The functions $g \in \mathscr{D}$ verifying

$$\nabla g(I_0) \neq 0 \quad , \qquad \mathtt{T}_{I_0}(g,r,n) \in \bigcup_{\substack{\mathtt{s}\in\mathbb{N}^{n-1}\\ 1\leq s_m\leq r-1\\ \forall m\in\{1,\dots,n-1\}}} \mathcal{P}(r,n)\backslash\boldsymbol{\Omega}_n^{r,\mathtt{s}}$$

are said to be steep at order r at the point I_0.

With respect to the original result by Nekhoroshev, a few aspects are refined or clarified in Theorem 4.1

1. The dependence of the steepness coefficients C_m, $m \in \{1,\dots,n-1\}$, and δ on the distance τ to the bad set $\boldsymbol{\Omega}_n^{r,\mathtt{s}}$, is made explicit. In particular, as it will be shown in section 7, for fixed $n, r, \mathtt{s}$, when $\tau \to 0$, then both $\delta \to 0$ and $C_m \to 0$ for all $m = 1,\dots,n-1$, whereas the bounds $\boldsymbol{\alpha}_m$ on the steepness indices are left unchanged. Hence, when $\tau \to 0$, steepness may "break down" due to the steepness coefficients tending to zero (whereas the steepness indices of those functions whose r-jet lies outside of $\boldsymbol{\Omega}_n^{r,\mathtt{s}}$ stay uniformly bounded).

   It is important to stress that the above reasonings do not necessarily imply that a function $g \in C_b^{2r-1}(\overline{B}^n(I_0,\varrho))$ whose r-jet satisfies $\mathtt{T}_{I_0}(g,r,n) \in \boldsymbol{\Omega}_n^{r,\mathtt{s}}$ - for some vector $\mathtt{s} \in \mathbb{N}^{n-1}$ as the one in Theorem A - is non-steep. For example, if for two vectors $\mathtt{s}, \mathtt{s}' \in \mathbb{N}^{n-1}$, $\mathtt{s}' \neq \mathtt{s}$, having the same properties of the one in Theorem A, one has $||\mathtt{T}_{I_0}(g,r,n) - \boldsymbol{\Omega}_n^{r,\mathtt{s}'}||_\infty > 0$, $||\mathtt{T}_{I_0}(g,r,n) - \boldsymbol{\Omega}_n^{r,\mathtt{s}}||_\infty = 0$, then g is steep at order r at I_0 for the vector $\mathtt{s}'$ but not for the vector $\mathtt{s}$. Hence, it admits different bounds on the steepness indices than the functions whose jets lie outside of $\boldsymbol{\Omega}_n^{r,\mathtt{s}}$. Therefore, the only result that one can infer from the relation $\mathtt{T}_{I_0}(g,r,n) \in \boldsymbol{\Omega}_n^{r,\mathtt{s}}$ is that, in case g were steep, its steepness indices would admit a different upper bound than the one in (4.1.1).

   By Theorem A, definition 4.1.1, and by the above discussion, the set

$$\bigcap_{\substack{\mathtt{s}\in\mathbb{N}^{n-1}\\ 1\leq s_m\leq r-1\\ \forall m\in\{1,\dots,n-1\}}} \boldsymbol{\Omega}_n^{r,\mathtt{s}} \subset \mathcal{P}(r,n) \tag{4.1.3}$$

   contains the r-jets of all C_b^{2r-1} functions around the non-critical point I_0 which are non-steep at order r at I_0. In the same way, when $r \to +\infty$, the Taylor coefficients of all non-steep analytic functions at the non-critical point I_0 belong to the set

$$\omega_n(I_0) := \bigcap_{r=2}^{\infty} \bigcap_{\substack{\mathtt{s}\in\mathbb{N}^{n-1} \\ 1\le s_m\le r-1 \\ \forall m\in\{1,\dots,n-1\}}} \mathbf{\Omega}_n^{r,\mathtt{s}} \quad \subset \quad \bigcup_{r=2}^{\infty} \mathcal{P}(r,n)\,. \tag{4.1.4}$$

Relation (4.1.4) is the explicit version of what Nekhoroshev meant when he wrote *"Hamiltonians which fail to be steep at a noncritical point are infinitely singular: they satisfy an infinite number of independent conditions on the Taylor coefficients"* (see [94], p. 426).

Actually, the strategy of proof given in the present work follows Nekhoroshev's reasonings by showing - for any given $Q \in \mathcal{P}(r,n)$- the existence of an arc, whose image is contained in the thalweg $\mathcal{T}_Q$ (see Definition 2.4.1), admitting a fitted parametrization whose derivatives are controlled by constants depending only on the number of variables n and on the degree r, but not on Q. This is a particular occurrence of the fact that - with the exception of small neighborhoods - a semi-algebraic set can be reparametrized by a collection of holomorphic functions with a domain of analyticity and an upper bound which depend only on the number of variables and on the number and on the degrees of the polynomials involved in the construction (see [116] and [118] for a two-dimensional semi-algebraic set, and [26], [50] for higher dimensional sets with more general properties than semi-algebraicness). The considered analytic reparametrization is a partial extension of a theorem (called algebraic lemma) due to Yomdin [115] and Gromov [67] which ensures that, for any semi-algebraic sets, there exists a collection of C^k-mappings which parametrize entirely the considered set. This latter theorem would be also relevant in our reasonings and, in theory, we would not have to exclude neighborhoods of the singularities as for the analytic reparametrizations (this causes extra difficulties in our proof). Actually, in our proof we must remove the singularities for other reasons and, also, the use of C^k-reparametrizations would not allow to obtain a characterization of non-steep functions as in 4.1.4, where a control of all the derivative up to infinity is needed.

2. The vector $\mathtt{s}$ does not appear in the original statement. Indeed, Nekhoroshev implicitly sets

$$s_m = \overline{s}_m := \begin{cases} \max\left\{1, r-1+m(n-m-1) - \dfrac{n(n-2)}{4}\right\} & , \qquad \text{for } n \text{ even} \\ \max\left\{1, r-1+m(n-m-1) - \dfrac{(n-1)^2}{4}\right\} & , \qquad \text{for } n \text{ odd}\ . \end{cases} \tag{4.1.5}$$

From a heuristic point of view, in ref. [96] this choice was probably conceived in the following way: in estimate (4.1.2), one may want to get rid of the quantity

$m(n-m-1)$, which attains the maximal value $n(n-2)/4$ for $m = n/2$ when n is even, and $(n-1)^2/4$ for $m = (n-1)/2$ when n is odd. Hence, (4.1.5) is the best choice which allows to get rid of the term $-m(n-m-1)$ in (4.1.2) and which still guarantees the essential condition $1 \le s_m \le r-1$.

Choice (4.1.5), in our case, yields (see (4.1.1)-(4.1.2))

$$\overline{\boldsymbol{\alpha}}_m(\overline{s}_m) := \begin{cases} \max\left\{1, r-1+m(n-m-1)-\dfrac{n(n-2)}{4}\right\}, & \text{for } n \text{ even}, m=1 \\ \max\left\{1, r-1+m(n-m-1)-\dfrac{(n-1)^2}{4}\right\}, & \text{for } n \text{ odd}, m=1 \\ \max\left\{1, 2r-3+2m(n-m-1)-\dfrac{n(n-2)}{2}\right\}, & \text{for } n \text{ even}, m\ge 2 \\ \max\left\{1, 2r-3+2m(n-m-1)-\dfrac{(n-1)^2}{2}\right\}, & \text{for } n \text{ odd}, m\ge 2\,, \end{cases} \tag{4.1.6}$$

and

$$\operatorname{codim} \boldsymbol{\Omega}_n^{r,\mathrm{s}} \ge \begin{cases} \max\left\{0, r-1-\dfrac{n(n-2)}{4}\right\}, & \text{if } n \text{ is even} \\ \max\left\{0, r-1-\dfrac{(n-1)^2}{4}\right\}, & \text{if } n \text{ is odd}\,. \end{cases} \tag{4.1.7}$$

For $m = 1$, the bound (4.1.6) on the steepness indices is half of the one found in [96]. For $m \ge 2$, the estimates in [96] coincide with (4.1.6). Estimate (4.1.7) on the codimension coincides with the one in [96].

3. Theorem 4.1 holds true even for functions in the class C_b^{r+1}. In that case, one considers $1 \le s_m \le \lfloor r/2 \rfloor$ for any $m = 1, \dots, n-1$, which yields worse estimates both on the indices of steepness and on the codimension. This is the case which was considered in [94], whereas the regularity C_b^{2r-1} was introduced in [96].

4.2 Explicit algebraic criteria for steepness

The kind of genericity stated in Theorem A implies the classic notions of genericity in topological and in measure sense. However, Theorem A alone is not sufficient when dealing with applications of Nekhoroshev's Theory to physical models. Infact, in order to infer long-time stability of a sufficiently regular integrable model Hamiltonian h under any sufficiently small and regular perturbation, one needs to have a criterion to check at which points of its domain the given function h is steep. As we shall show in the sequel, establishing a criterion of this kind is a non-trivial developement of the proof of the genericity of steepness. Namely, in the rest of this section we will present explicit algebraic criteria for steepness which involve the Taylor coefficients at any order of the studied function.

In order to give rigorous statements, we first need to introduce some notations.

4.2.1 Some additional notations

For any positive integers $n \geq 2$ and $k \in \{2, \dots, n\}$, we introduce the notation

$$\mathscr{V}^1(k,n) := \{(v, u_2, \dots, u_k) \in \mathbb{R}^n \times \mathbb{U}(k-1,n) |\ \mathrm{rk}(v, u_2, \dots, u_k) = k\} \ . \tag{4.2.1}$$

For fixed integers $m \geq 2$ and $\alpha \geq 0$, for any $\beta \in \{0, ..., \alpha\}$, and for any $i \in \{1, ..., m\}$, we also introduce the family of multi-indices

$$\mathbb{N}^m \ni \nu(i,\beta) := \begin{cases} (\beta+1, 0, ..., 0) \ , & \text{for } i = 1 \\ (\beta, 0, ..., 0, 1, 0, ..., 0) \ , & \text{for } i = 2, ..., m \end{cases} \tag{4.2.2}$$

where the "1" fills the i-th slot for $i = 2, ..., m$. When $\alpha \geq 1$, we denote the multi-indices $\mu \in \mathbb{N}^m$ of length $2 \leq |\mu| := ||\mu||_1 \leq \alpha + 1$ not belonging to the family (4.2.2) with

$$\mathcal{M}_m(\alpha) := \{\mu \in \mathbb{N}^m \ , \ \ 2 \leq |\mu| \leq \alpha + 1\} \setminus \bigcup_{\substack{i=1,...,m \\ \beta=1,...,\alpha}} \{\nu(i,\beta)\} \ . \tag{4.2.3}$$

Moreover, for given values of $\alpha \in \mathbb{N}$, $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{N}^m$ and $\ell \in \{1, ..., m\}$, we introduce the multi-index $\widetilde{\mu}(\ell) = (\widetilde{\mu}_1(\ell), \dots, \widetilde{\mu}_m(\ell))$ for which $\widetilde{\mu}_i(\ell) := \mu_i - \delta_{i\ell}$, $i = 1, \dots, m$, and the set

$$\begin{aligned} \mathcal{G}_m(\widetilde{\mu}(\ell), \alpha) := \Bigg\{ & k := (k_{22}, \dots, k_{2\alpha}, \dots, k_{m2}, \dots, k_{m\alpha}) \in \mathbb{N}^{(m-1)\times(\alpha-1)} : \\ & \sum_{i=2}^{\alpha} k_{ji} = \widetilde{\mu}_j(\ell) \ , \ \ \widetilde{\mu}_1(\ell) + \sum_{j=2}^{m} \sum_{i=2}^{\alpha} i \ k_{ji} = \alpha \Bigg\} \ . \end{aligned} \tag{4.2.4}$$

For any $k \in \mathcal{G}_m(\widetilde{\mu}(\ell), \alpha)$, we set $k! := k_{22}! \dots k_{2\alpha}! \dots k_{m2}! \dots k_{m\alpha}!$ and for any $\mu \in \mathbb{N}^m$ we set $\mu! := \mu_1! \dots \mu_m!$.

We define also

$$\mathcal{E}_m(\ell, \alpha) := \{\mu \in \mathbb{N}^m \mid \mathcal{G}_m(\widetilde{\mu}(\ell), \alpha) \neq \emptyset\} \ . \tag{4.2.5}$$

Finally, we consider a quadruplet of positive integers $r \geq 2, n \geq 3$, $1 \leq s \leq r-1$, $2 \leq m \leq n-1$, a point $I_0 \in \mathbb{R}^n$, and a function h of order C^r around I_0.

With this setting, for any given $\alpha \in \{1, \dots, s\}$, and $\ell \in \{1, \dots, m\}$, we introduce the functions

$$\mathcal{H}^{h,I_0}_{m,\ell,\alpha} : \mathscr{V}^1(m,n) \times \mathbb{R}^{(m-1)s} \to \mathbb{R}$$

associating to any element $(v, u_2, \dots, u_m) \in \mathscr{V}^1(m,n)$ and to any vector $a(m,s) =$

$(a_{21}, \dots, a_{2s}, \dots, a_{m1}, \dots, a_{ms}) \in \mathbb{R}^{(m-1)\times s}$ the following quantities

$$\mathcal{H}^{h,I_0}_{m,1,1}(v,u_2,\dots,u_m,a(m,s)) := h^2_{I_0}[v,v] \quad , \qquad \alpha = 1\ , \ \ell = 1$$
$$\mathcal{H}^{h,I_0}_{m,\ell,1}(v,u_2,\dots,u_m,a(m,s)) := h^2_{I_0}\left[v,u_\ell\right] \quad , \qquad \alpha = 1\ , \ \ell \in \{2,\dots,m\}$$
$$\mathcal{H}^{h,I_0}_{m,1,2}(v,u_2,\dots,u_m,a(m,s)) := h^3_{I_0}[v,v,v] \quad , \qquad \text{if } s \geq 2\ , \ \text{ for } \alpha = 2\ , \text{ and } \ell = 1$$

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if $s \geq 3$, for $\alpha \in \{3,\dots,s\}$, and $\ell = 1$

$$\mathcal{H}^{h,I_0}_{m,1,\alpha}(v,u_2,\dots,u_m,a(m,s)) :=$$
$$\frac{1}{\alpha!}h^{\alpha+1}_{I_0}\left[v,\dots,v\right] + \sum_{\beta=1}^{\alpha-1}\frac{1}{(\beta-1)!}h^{\beta+1}_{I_0}\Big[\overbrace{\ v\ }^{\beta},\sum_{i=2}^{m}a_{i(\alpha-(\beta-1))}\,u_i\Big]$$
$$+\sum_{\substack{\mu\in\mathcal{E}_m(1,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_1\neq 0}}\ \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(1),\alpha)}\frac{h^{|\mu|}_{I_0}\Big[\overbrace{\ v\ }^{\mu_1-1},\overbrace{a_{22}u_2}^{k_{22}},\dots,\overbrace{a_{2\alpha}u_2}^{k_{2\alpha}},\dots,\overbrace{a_{m2}u_m}^{k_{m2}},\dots,\overbrace{a_{m\alpha}u_m}^{k_{m\alpha}},v\Big]}{(\mu_1-1)!\,k!} \tag{4.2.6}$$

if $s \geq 2$, for $\alpha \in \{2,\dots,s\}$, and $\ell \in \{2,\dots,m\}$

$$\mathcal{H}^{h,I_0}_{m,\ell,\alpha}(v,u_2,\dots,u_m,a(m,s)) := \frac{1}{\alpha!}h^{\alpha+1}_{I_0}\Big[\overbrace{\ v\ }^{\alpha},u_\ell\Big]$$
$$+\sum_{\substack{\mu\in\mathcal{E}_m(\ell,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_\ell\neq 0}}\ \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)}\frac{h^{|\mu|}_{I_0}\Big[\overbrace{\ v\ }^{\mu_1},\overbrace{a_{22}u_2}^{k_{22}},\dots,\overbrace{a_{2\alpha}u_2}^{k_{2\alpha}},\dots,\overbrace{a_{m2}u_m}^{k_{m2}},\dots,\overbrace{a_{m\alpha}u_m}^{k_{m\alpha}},u_\ell\Big]}{\mu_1!\,k!}. \tag{4.2.7}$$

With the setting above, we can state the first explicit criterion for steepness. Its Corollary B2 is a refined version of Theorem 2.3.3.

4.2.2 Theorem B and related corollaries

Theorem (B). *Let $r, n \geq 2$ be two integers, and let $\mathtt{s} := (s_1,\dots,s_{n-1}) \in \mathbb{N}^{n-1}$ be a vector verifying $1 \leq s_i \leq r-1$ for all $i = 1,\dots,n-1$. Consider a point $I_0 \in \mathbb{R}^n$, a real number $\varrho > 0$, and a function h of class $C^{2r-1}_b(B^n(I_0,\varrho))$ verifying $\nabla h(I_0) \neq 0$.*

i) If the system

$$\begin{cases} w \in \mathbb{S}^n \\ h^1_{I_0}[w] = h^2_{I_0}[w,w] = \cdots = h^{s_1+1}_{I_0}[w,\dots,w] = 0 \end{cases} \tag{4.2.8}$$

has no solution[1]*, then h is steep around the point I_0 on the affine subspaces of dimension one, with steepness index bounded by s_1.*

In the sequel, we set $N = N(r,n) := \dim \mathcal{P}(r,n)$.

ii) If, for some $m \in \{2,\dots,n-1\}$, there exists $R_m > 0$ such that for any polynomial $S \in B^N(\mathbb{T}_{I_0}(h,r,n), R_m)$ the system

$$\begin{cases} (u_1,\dots,u_m) \in \mathbb{U}(m,n) \\ a(m,s_m) := (a_{21},\dots,a_{2s_m},\dots,a_{m1},\dots,a_{ms_m}) \in \mathbb{R}^{(m-1)\times s_m} \\ v = u_1 + \sum_{j=2}^m a_{j1}\, u_j \\ S^1_{I_0}[v] = S^1_{I_0}[u_2] = \cdots = S^1_{I_0}[u_m] = 0 \\ \sum_{\ell=1}^m \sum_{\alpha=1}^{s_m} \left| \mathcal{H}^{S,I_0}_{m,\ell,\alpha}(v,u_2,\dots,u_m,a(m,s_m)) \right| = 0 \end{cases} \tag{4.2.9}$$

has no solution, then h is steep in a neighborhood of I_0 on the affine subspaces of dimension m, with steepness index bounded by $\boldsymbol{\alpha}_m \le 2s_m - 1$.

Though the quantities which are involved in Theorem B are quite cumbersome, the idea behind the result is not difficult to grasp: condition (4.2.9) amounts to asking that the r-jet of $h(I) - h(I_0)$ lies outside of the semi-algebraic set $\boldsymbol{\Omega}^{r,\mathrm{s}}_n$ defined in Theorem A. This will be made clearer in Corollary B2.

As it will be discussed in the technical sections of the present work, for any $m \in \{2,\dots,n-1\}$, the real parameters $a_{21},\dots,a_{2s_m},\dots,a_{m1},\dots,a_{ms_m}$ appearing in (4.2.6)-(4.2.7) and in the statement of Theorem B represent the Taylor coefficients of analytic curves of the type

$$\gamma(t) := \Big\{ x_1(t) := t \quad , \qquad x_j(t) := \sum_{i=1}^{+\infty} a_{ji} t^i \qquad j = 2,\dots,m \tag{4.2.10}$$

which, for any m-dimensional affine subspace $I_0 + \Gamma^m$, contain the locus of minima of the projection $||\pi_{\Gamma^m} \nabla \mathbb{T}_{I_0}(h,r,n)||_2$. For any given Γ^m and for any function h regular around I_0, the existence of a minimal curve of the form (4.2.10) is ensured by Theorem 5.0.1 in the sequel.

Theorem B comes together with important corollaries.

The following one is well-known: its statement can be found in [95], whereas its proof can be found in [47]. As we shall see, in our context it is a simple consequence of Theorem B.

[1] In this case, h is said to be $s_1 + 1$-jet non-degenerate at the origin.

Corollary (B1). *Consider an integer $n \geq 2$, a point $I_0 \in \mathbb{R}^n$, and a function h of class C^5 around I_0, satisfying $\nabla h(I_0) \neq 0$. If the system*

$$\begin{cases} w \in \mathbb{S}^n \\ h^1_{I_0}[w] = h^2_{I_0}[w,w] = h^3_{I_0}[w,w,w] = 0 \qquad \text{(3-jet non degeneracy)} \end{cases} \tag{4.2.11}$$

has no solution, then h is steep in a neighborhood of I_0, and its indices satisfy

$$\alpha_1 = 2 \quad , \qquad \max_{m=2,\dots,n-1}\{\alpha_m\} \leq 3 \, .$$

Remark 4.2.1. Actually, as a more careful analysis of 3-jet non-degenerate functions shows, the result is true for C^4 functions and one can take $\max_{m=1,\dots,n-1}\{\alpha_m\} \leq 2$ (see [47]).

Thanks to Theorem B, moreover, we have a more explicit characterisation of the sets $\mathbf{\Omega}^{r,\mathtt{s}}_n$ appearing in the statement of Theorem A. Namely, as it was the case in the hypotheses of Theorem B, we consider two integers $r, n \geq 2$, and a vector $\mathtt{s} := (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$, with $1 \leq s_i \leq r-1$ for all $i = 1, \dots, n-1$. Also, we take a point $I_0 \in \mathbb{R}^n$. This time, differently to what we did in Theorem B, we do not consider a fixed function.

Corollary (B2). *For $n \geq 2$, and $m = 1$, we indicate by $\mathcal{Z}^{r,s_1,1}_n$ the algebraic set of $\mathcal{P}^\star(r,n) \times \mathbb{S}^n$ determined by*

$$\begin{cases} w \in \mathbb{S}^n \quad , \qquad P \in \mathcal{P}^\star(r,n) \\ P^1_{I_0}[w] = P^2_{I_0}[w,w] = \dots = P^{s_1+1}_{I_0}[w,\dots,w] = 0 \, . \end{cases} \tag{4.2.12}$$

For $n \geq 3$, and for any given $m \in \{2, \dots, n-1\}$, we denote by $\mathcal{Z}^{r,s_m,m}_n$ the algebraic set of $\mathcal{P}^\star(r,n) \times \mathbb{R}^{(m-1)s_m} \times \mathscr{V}^1(m,n)$ determined by

$$\begin{cases} (u_1, \dots, u_m) \in \mathbb{U}(m,n) \quad , \qquad P \in \mathcal{P}^\star(r,n) \\ a(m,s_m) := (a_{21}, \dots, a_{2s_m}, \dots, a_{m1}, \dots, a_{ms_m}) \in \mathbb{R}^{(m-1)\times s_m} \\ v = u_1 + \sum_{j=2}^m a_{j1}\, u_j \\ \sum_{\ell=1}^m \sum_{\alpha=1}^{s_m} \left| \mathcal{H}^{P,I_0}_{m,\ell,\alpha}(v, u_2, \dots, u_m, a(m,s_m)) \right| = 0 \, . \end{cases} \tag{4.2.13}$$

With this setting, one has

$$\bigcup_{m=1}^{n-1} \text{closure} \left(\Pi_{\mathcal{P}^\star(r,n)} \, \mathcal{Z}^{r,s_m,m}_n \right) = \mathbf{\Omega}^{r,\mathtt{s}}_n \, , \tag{4.2.14}$$

where $\mathbf{\Omega}^{r,\mathtt{s}}_n$ is the semi-algebraic set introduced in Theorem A.

Remark 4.2.2. Since the sets $\mathcal{Z}^{r,s_m,m}_n$, $m = 1, \dots, n-1$ in Corollary 4.2.2 are algebraic, the Theorem of Tarski and Seidenberg (see Th. A.1.1) - together with expression (4.2.14) and Proposition A.1.2 - assures that $\mathbf{\Omega}^{r,\mathtt{s}}_n$ is a semi-algebraic set of $\mathcal{P}(r,n)$, as we already knew by Theorem 4.1. Moreover, it is worth to notice that - at least in principle - one could find the explicit expression for $\mathbf{\Omega}_n^{r,\mathrm{s}}$. Infact, the Theorem of Tarski and Seidenberg is somewhat "constructive", in the sense that there exist algorithms that allow to find the explicit expression for the projection and the closure of any semi-algebraic set (see e.g. [18]). However, these general algorithms are not very useful in applications, as their complexity grows double-exponentially with the number of the involved variables (see [72]). As we shall show in Theorems C1-C2 below, in "most cases" (in a sense that will be clarified in Theorem C3) the sets $\mathcal{Z}_n^{r,s_m,m}$, with $m = 2, \dots, n-1$ can be projected onto $\mathcal{P}^\star(r,n) \times \mathscr{V}^1(m,n)$ with the help of a simple algorithm involving only linear operations. Moreover, such a projection yields a closed semi-algebraic set of $\mathcal{P}^\star(r,n) \times \mathscr{V}^1(m,n)$. This implies a further criterion to check steepness of a given function.

Finally, using Theorem B we can state a sufficient condition for non-steepness at a given point, namely

Corollary (B3). *Consider a point $I_0 \in \mathbb{R}^n$, and a function h in the real-analytic class around I_0 verifying $\nabla h(I_0) \neq 0$.*

If at least one of the two following conditions is satisfied, then h is non-steep at I_0.

1. *There exists $w \in \mathbb{S}^n$ such that $h^r_{I_0}[w] = 0\ ,\ \ \forall r \in \mathbb{N}$.*
2. *For some $m \in \{2, \dots, n-1\}$, there exist*
   - *(a) $m-1$ real sequences $\{a_{ji}\}_{i\in\mathbb{N}}$, $j = 2, \dots, m$ and a number $\mathtt{t} > 0$ such that the expansions $\sum_{i=1}^{+\infty} a_{ji} t^i$ admit a radius of convergence greater than $\mathtt{t}$ for all m ;*
   - *(b) m linearly-independent vectors $v, u_2, \dots, u_m \in \mathscr{V}^1(m,n)$;*

   such that for all integer $r \geq 2$ the following system is satisfied:

$$\begin{cases} (u_1, \dots, u_m) \in \mathbb{U}(m,n) \\ a(m, r-1) := (a_{21}, \dots, a_{2(r-1)}, \dots, a_{m1}, \dots, a_{m(r-1)}) \in \mathbb{R}^{(m-1)\times(r-1)} \\ v = u_1 + \sum_{j=2}^m a_{j1}\, u_j \\ h^1_{I_0}[v] = h^1_{I_0}[u_2] = \dots = h^1_{I_0}[u_m] = 0 \\ \sum_{\ell=1}^m \sum_{\alpha=1}^{r-1} \left| \mathcal{H}^{P,I_0}_{m,\ell,\alpha}(v, u_2, \dots, u_m, a(m, s_m)) \right| = 0\ . \end{cases} \tag{4.2.15}$$

Remark 4.2.3. Since we consider any $r \geq 2$, we have an infinite system.

4.2.3 Theorems C1-C2-C3

As we have showed above, Theorem B constitutes an explicit criterion for steepness which, however, for any given value of $n \geq 3$, $m \in \{2, \dots, n-1\}$ and $s_m \in \{1, \dots, r-1\}$

depends on the additional parameters $a_{21}, \dots, a_{2s_m}, \dots, a_{m1}, \dots, a_{ms_m} \in \mathbb{R}^{(m-1)s_m}$ and on the vectors $v, u_2, \dots, u_m \in \mathscr{V}^1(m,n)$. As we have already pointed out in Remark 4.2.2, it is possible in principle to reduce these quantities from system (4.2.9), by the means of classical algorithms of semi-algebraic geometry (see [18]). However, in general the complexity of the latter grows double exponentially in the number of variables (see [72]) making them of little use in practice.

However, since the quantities in (4.2.6)-(4.2.7) are explicit, one may attempt to exploit their specific form in order to find an algorithm which is simpler than the classic ones and that allows to eliminate at least the parameters $a_{22}, \dots, a_{2s_m}, \dots, a_{m2}, \dots, a_{ms_m}$ from system (4.2.9). In this way, one would have an explicit criterion for steepness involving only the multilinear forms of the tested function h up to a given order, the parameters $a_{21}, \dots, a_{m1}$, and the vectors $v, u_2, \dots, u_m$.

Having an explicit criterion for steepness involving only the coefficients $a_{21}, \dots, a_{m1}$ and the vectors $v, u_2, \dots, u_m$ as additional free parameters constitutes a qualititative improvement w.r.t. Theorem B in view of possible applications. Infact, as we shall show in the sequel, without any loss of generality the numbers $a_{21}, \dots, a_{m1}$ and the vector v can be assumed to belong to a compact subset. Moreover, the vectors $u_2, \dots, u_m$ belong to $\mathtt{U}(n, m-1)$, which is compact by definition. Therefore, if one manages to find a criterion that does not involve $a_{22}, \dots, a_{2s_m}, \dots, a_{m2}, \dots, a_{ms_m}$, one only has to deal with additional parameters belonging to a compactum. Moreover, the presence of the vectors $v, u_2, \dots, u_m$ permits to keep track of the subspaces one is working on; namely, it is possible to isolate the subspaces where the studied function is non-steep.

As we prove in sections 9-10, for a generic regular test function h and for any $m \in \{2, \dots, n\}$, on most of the m-dimensional subspaces of the Grassmannian $\mathtt{G}(m,n)$ one is able to apply an explicit criterion to check steepness that does not involve the parameters $a_{22}, \dots, a_{2s_m}, \dots, a_{m2}, \dots, a_{ms_m}$: this is the content of Theorems C1-C2-C3.

In order to state these results, we start by considering an integer $n \geq 3$, and a function h of class C^2 around the origin, satisfying $\nabla h(0) \neq 0$. Now, for any $m \in \{2, \dots, n-1\}$ we need to consider three subsets of the Grassmannian $\mathtt{G}(m,n)$.

Definition 4.2.1. For any integer $m \in \{2, ..., n-1\}$ and any integer $j \in \{0, 1\}$, we indicate by $\Lambda_j(h, m, n)$ the subset of $\mathtt{G}(m,n)$ containing those m-dimensional subspaces Γ^m satisfying

1. $\nabla h(0) \perp \Gamma^m$;

2. the Hessian matrix of the restriction of h to Γ^m, calculated at the origin, has exactly j null eigenvalues.

Definition 4.2.2. For any fixed $m \in \{2, ..., n-1\}$, the symbol $\Lambda_2(h, m, n)$ indicates the subset of those m-dimensional linear subspaces $\Gamma^m \in \mathtt{G}(m,n)$ that satisfy the following conditions

1. $\nabla h(0) \perp \Gamma^m$;

2. the Hessian matrix of the restriction of h to Γ^m, calculated at the origin, has 2 or more null eigenvalues.

With the above setup, for any fixed $m \in \{2, \dots, n-1\}$, we have

$$\{\Gamma^m \in \mathbb{G}(m,n) | \Gamma^m \perp \nabla h(0)\} = \Lambda_0(h,m,n) \bigsqcup \Lambda_1(h,m,n) \bigsqcup \Lambda_2(h,m,n) \,. \quad (4.2.16)$$

We are now ready to state Theorems C1-C2-C3.

Consider two positive integers $r, n \geq 2$, a vector $\mathtt{s} := (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$, with $1 \leq s_i \leq r-1$ for all $i = 1, \dots, n-1$, and a function h of class C_b^{2r-1} around the origin, satisfying $\nabla h(0) \neq 0$. Then, for any given $m \in \{2, ..., n-1\}$, one has the following results (which, considered together, are refined versions of Theorems 2.3.4-2.3.5 in the introduction):

Theorem (C1). *h is steep at the origin, with index $\alpha_m = 1$, on the m-dimensional subspaces belonging to $\Lambda_0(h,m,n)$.*

Theorem (C2). *If $s_m \geq 2$ there exist two semi-algebraic sets*

$$\mathscr{A}_1(r, s_m, n, m), \mathscr{A}_2(r, s_m, n, m) \subset \mathcal{P}^\star(r,n) \times \mathbb{R}^{m-1} \times \mathscr{V}^1(m,n)$$

satisfying the following properties:

1. *The form of $\mathscr{A}_1(r, s_m, n, m)$ can be explicitly computed starting from the expression of set $\mathcal{Z}_n^{r,s_m,m}$ in (4.2.13) by the means of an algorithm involving only linear operations.*

2. *If system*

$$\begin{cases} (u_1, \dots, u_m) \in \mathbb{U}(m,n) \quad , \qquad \textit{Span}\ (u_1, u_2, \dots, u_m) \in \Lambda_1(h,m,n) \\ (\mathtt{T}_0(h,r,n), 0, u_1, u_2, \dots, u_m) \in \mathscr{A}_1(r, s_m, n, m) \end{cases} \quad (4.2.17)$$

   has no solution, then h is steep around the origin with index $\alpha_m \leq 2s_m - 1$ on any subspace $\Gamma^m \in \Lambda_1(h,m,n)$.

3. *There exists a positive constant $\mathscr{K} = \mathscr{K}(r,n,m)$ such that if*

$$\begin{cases} (a_{21}, \dots, a_{m1}) \in \overline{B}^{m-1}(\mathscr{K}) \\ (u_1, \dots, u_m) \in \mathbb{U}(m,n) \quad , \qquad v := u_1 + \sum_{i=2}^m a_{i1} u_i \\ \textit{Span}\ (v, u_2, \dots, u_m) \in \Lambda_2(h,m,n) \\ (\mathtt{T}_0(h,r,n), a_{21}, \dots, a_{m1}, u_1, u_2, \dots, u_m) \in \mathscr{A}_2(r, s_m, n, m) \end{cases} \quad (4.2.18)$$

   has no solution, then h is steep around the origin with index $\alpha_m \leq 2s_m - 1$ on any subspace $\Gamma^m \in \Lambda_2(h,m,n)$.

Remark 4.2.4. We observe that the statement above gives no information about the explicit expression of subset $\mathscr{A}_2(r, s_m, n, m)$. As it will be shown in sections 9-10, the linear algorithm used to deduce the form of $\mathscr{A}_1(r, s_m, n, m)$ starting from set $\mathcal{Z}_n^{r,s_m,m}$ in (4.2.13) fails in case $v, u_2, \ldots, u_m$ span a subspace belonging to $\Lambda_2(h, m, n)$. Therefore, in order to find the explicit expression for $\mathscr{A}_2(r, s_m, n, m)$ one is obliged to apply the classical, much slower algorithms of real-algebraic geometry to the set $\mathcal{Z}_n^{r,s_m,m}$ determined by system (4.2.13). Thus, checking steepness on the subspaces of $\Lambda_2(h, m, n)$ is more complicated than on those belonging to $\Lambda_1(h, m, n)$, as one is obliged either to apply slow algorithms to find the explicit expression of system (4.2.18), or to use the statement of Theorem B, which nevertheless depends on the non-compact real coefficients $a_{22}, \ldots, a_{2s_m}, \ldots, a_{m2}, \ldots, a_{ms_m}$.

However, for a generic function h the subsets $\Lambda_1(h, m, n)$ and $\Lambda_2(h, m, n)$ are "rare" inside the Grassmannian $\mathtt{G}(m, n)$. Namely, in Theorem C3 below we prove that for any $m \in \{2, \ldots, n-1\}$ and for any bilinear symmetric non-degenerate form $\mathsf{B} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, the subspaces of dimension m on which the restriction of B has one or two null eigenvalues are rare in $\mathtt{G}(m, n)$, both in measure and in topological sense.

Theorem (C3). *Let* $\mathsf{B} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a bilinear, symmetric, nondegenerate form, and let* $m \in \{2, \ldots, n-1\}$ *be a positive integer.*

For $j \in \{1, 2\}$*, denote by* $\mathtt{G}_j(\mathsf{B}, m, n) \subset \mathtt{G}(m, n)$ *the subset of linear m-dimensional subspaces on which the restriction of* B *has at least j null eigenvalues.*

Then

1. $\mathtt{G}_1(\mathsf{B}, m, n)$ *is contained in a submanifold of codimension one in* $\mathtt{G}(m, n)$*;*

2. $\mathtt{G}_2(\mathsf{B}, m, n)$ *is obtained by the intersection of* $\mathtt{G}_1(m, n)$ *with another subset contained in a submanifold of codimension one in* $\mathtt{G}(m, n)$*.*

Finally, we state the following conjecture, which will hopefully be proved in a future work.

Conjecture: for a generic bilinear form B, the subset $\mathtt{G}_2(\mathsf{B}, m, n)$ appearing in Theorem C3 is contained in a submanifold of codimension two in $\mathtt{G}(m, n)$.

Chapter 5

The Thalweg and its properties

It is clear from Definition 11.2.1 that studying the steepness property at the origin of a given function $h \in C^2(B^n(0, 2\delta), \mathbb{R})$ satisfying $\nabla h(0) \neq 0$, amounts to studying the projection of its gradient on any m-dimensional subspace Γ^m perpendicular to $\nabla h(0)$, with $m \in \{1, ..., n-1\}$. More precisely, given $\delta > 0$, for any fixed $\eta \in [0, \delta]$ we are interested in the quantity

$$\mu_h(\Gamma^m, \eta) := \min_{u \in \Gamma^m,\, ||u|| = \eta} ||\pi_{\Gamma^m} \nabla h(u)||_2 \, .$$

Since, for any given Γ^m orthogonal to $\nabla h(0)$ and for any $\eta \in [0, \delta]$, the value $\mu_h(\Gamma^m, \eta)$ is attained at some point of the m-dimensional sphere

$$\mathcal{S}^m_\eta := \{u \in \Gamma^m \mid ||u||_2 = \eta\} \, ,$$

it makes sense to give the following

Definition 5.0.1. We call *Thalweg* of h on Γ^m the set

$$\mathcal{T}(h, \Gamma^m) := \{I^\star \in \Gamma^m : ||\pi_{\Gamma^m} \nabla h(I^\star)|| = \mu_h(\Gamma^m, \eta) \quad \text{for} \quad 0 \leq \eta := ||I^\star|| \leq \delta\} \, .$$

In the sequel, we will be interested in studying the thalweg $\mathcal{T}(\mathtt{T}_0(h, r, n), \Gamma^m)$ of the Taylor polynomial $\mathtt{T}_0(h, r, n)$. Namely, the goal of this section is to prove the following

Theorem 5.0.1. *(Nekhoroshev, [94]) For any pair of integers $r, n \geq 2$, and for any real $\delta > 0$, consider a function $h \in C^r(B^n(0, 2\delta))$ satisfying $\nabla h(0) \neq 0$. Then, given a number $m \in \{1, ..., n-1\}$, for any m-dimensional subspace Γ^m orthogonal to $\nabla h(0)$ there exists a semi-algebraic curve γ with values in $\mathcal{T}(\mathtt{T}_0(h, r, n), \Gamma^m)$ such that $\gamma(0) = 0$ and*

1. *For any fixed $\eta \geq 0$, the intersection $Im(\gamma) \cap \mathcal{S}^m_\eta$ is a singleton;*
2. *There exists a positive integer $\mathtt{d} = \mathtt{d}(r, n, m)$ that bounds the diagram (see Def. A.1.2) of graph(γ);*

3. *There exists* $\mathtt{K} = \mathtt{K}(r, n, m) > 1$ *such that, for any* $\lambda > 0$*, the curve* γ *is real-analytic on some closed interval* $\mathtt{I}_\lambda \subset [-\lambda, \lambda]$ *of length* $\lambda/\mathtt{K}$*, with complex analyticity width* $\lambda/\mathtt{K}$*;*

4. *Over* $\mathtt{I}_\lambda$*,* γ *is an i-arc, i.e. it can be parametrized by*

$$\gamma(t) := \begin{cases} x_i(t) = t & \textit{for some } i \in \{1, ..., m\} \\ x_j(t) = f_j(t) & \textit{for all } j \in \{1, ..., m\},\ j \neq i \end{cases} \qquad t \in \mathtt{I}_\lambda$$

where the $f_j(t)$ *are Nash (i.e. analytic-algebraic) functions;*

5. γ *satisfies a Bernstein's inequality on its Taylor coefficients over the interval* $\mathtt{I}_\lambda$*. Namely, indicating by*

$$f_j(t) = \sum_{\beta=0}^{+\infty} a_{j\beta}(u)\, t^\beta \quad , \qquad j \in \{1, ..., m\}\ ,\ \ j \neq i\ ,$$

the Taylor expansion of f_j *at some point* $u \in \mathtt{I}_\lambda$*, there exists a positive constants* $K_2(r, n, m)$*, and* $\mathtt{M} = \mathtt{M}(r, n, m, \beta) := \beta! \times \mathtt{K}^\beta \times K_2(r, n, m)$ *for which the following uniform estimate holds:*

$$\max_{u \in \mathtt{I}_\lambda} |a_{j\beta}(u)| \leq \frac{\mathtt{M}}{\lambda^{\beta-1}}\ . \tag{5.0.1}$$

Remark 5.0.1. The Theorem above corresponds to reasonings holding in the polynomial setting. Moreover, the constants $\mathtt{d}, \mathtt{K}, \mathtt{M}$ depend only on the degree of the considered polynomial. Consequently, this Theorem holds uniformly for any r-jet of any function $h \in C^r(B^n(0, 2\delta), \mathbb{R})$.

Remark 5.0.2. Nekhoroshev calls γ "minimal arc with uniform characteristics" (see [94], section 4). In that work, the statement of Theorem 5.0.1 is not given in the form above but is rather split in dispersed parts. Moreover, many of the modern tools of real-algebraic geometry were lacking at that time, so that the redaction of his work appears quite obscure in some parts. These two elements makes difficult for the reader to reconstruct simply Theorem 5.0.1 from Nekhoroshev's original paper.

Remark 5.0.3. The Bernstein's inequality at point 5 of Theorem 5.0.1 is essential in order to have stable[1] lower estimates for the steepness coefficients of h. For more details about this result, which is interesting in itself and has applications in various fields of mathematics, see refs. [107] and [16].

Some intermediate Lemmas are needed before demonstrating Theorem 5.0.1.

Lemma 5.0.1. *Take any triplet of integers* $r, n \geq 2$ *and* $m \in \{1, ..., n-1\}$*. There exists* $\mathsf{d} = \mathsf{d}(r, n, m) \geq 0$ *such that, for any* $Q \in \mathcal{P}(r, n)$ *satisfying* $\nabla Q(0) \neq 0$*, and for any subspace* Γ^m *perpendicular to* $\nabla Q(0)$*, the thalweg* $\mathcal{T}(Q, \Gamma^m)$ *is a semi-algebraic set satisfying* $\operatorname{diag}(\mathcal{T}(Q, \Gamma^m)) \leq \mathsf{d}$ *(see Def. A.1.2). Moreover, for any fixed* $\eta_0 > 0$*, the intersection of* $\mathcal{T}(Q, \Gamma^m)$ *with the sphere* $\mathcal{S}^m_{\eta_0} \subset \Gamma^m$ *is compact.*

[1] In the sense given in Th. 4.1, that is valid for an open set of functions.

Proof. Γ^m is obviously isomorphic to $\mathbb{R}^m$ and admits a global system of orthonormal coordinates $x = (x_1, ..., x_m)$. We denote by $P(x) \in \mathcal{P}(r, m)$ the restriction of $Q(I)$ to $\Gamma^m \simeq \mathbb{R}^m$. Since we endow Γ^m with the induced euclidean metric, studying the norm of the projection of $\nabla_I Q(I)$ on Γ^m amounts to studying the induced norm of $\nabla_x P(x)$ on $\Gamma^m \simeq \mathbb{R}^m$. Now, consider the semi-algebraic set

$$\mathcal{E} := \{(x, y, \eta) \in \mathbb{R}^{2m} \times \mathbb{R} \,:\, ||x||_2^2 = ||y||_2^2 = \eta^2, \eta > 0, ||\nabla P(x)||_2 > ||\nabla P(y)||_2\} \tag{5.0.2}$$

By the Theorem of Tarski and Seidenberg A.1.1 and Proposition A.1.1, we have that the set $\mathbb{R}^m \backslash \pi_x \mathcal{E} := \{x \in \mathbb{R}^m \times \mathbb{R} \,:\, \forall (y, \eta) \in \mathbb{R}^m \times \mathbb{R}, (x, y, \eta) \notin \mathcal{E}\}$ is semi-algebraic. We claim that it coincides with $\mathcal{T}(Q, \Gamma^m)$. Infact, by the definition of $\mathcal{E}$, it is clear that for any given $\eta > 0$ one has

$$x \in \mathbb{R}^m \backslash \pi_x \mathcal{E} \iff ||\nabla P(x)||_2 \leq ||\nabla P(y)||_2 \text{ for all } y \in \mathbb{R}^m \text{ s.t. } ||y||_2{}^2 = ||x||_2{}^2 \,, \tag{5.0.3}$$

so that $x \in \mathbb{R}^m \backslash \pi_x \mathcal{E}$ is the locus of minima on any given sphere for $||\nabla P||_2$ (that is for $||\pi_{\Gamma^m} \nabla Q||_2$), that is it coincides with the Thalweg $\mathcal{T}(Q, \Gamma^m)$. Moreover, since $\deg P \leq r$, the diagram of $\mathcal{E}$ is uniformly bounded w.r.t. any $P \in \mathcal{P}(r, m)$ and, again by the Theorem of Tarski and Seidenberg, the same is true for $\pi_x \mathcal{E}$ and for $\mathbb{R}^m \backslash \pi_x \mathcal{E} \equiv \mathcal{T}(Q, \Gamma^m)$.

It remains to prove that $\mathcal{T}(Q, \Gamma^m) \cap \mathcal{S}^m_{\eta_0}$ is compact. By construction, $\mathcal{T}(Q, \Gamma^m) \cap \mathcal{S}^m_{\eta_0}$ is the locus of minima of $||\nabla P||$ on $\mathcal{S}^m_{\eta_0}$. Since the function $||\nabla P||_2$ is continuous on $\mathcal{S}^m_{\eta_0}$, the inverse image of its minimal value on $\mathcal{S}^m_{\eta_0}$ is closed. Since $\mathcal{S}^m_{\eta_0}$ is compact, the thesis follows. □

The next Lemma shows how an analytic curve with uniform characteristics can be extracted from the Thalweg.

Lemma 5.0.2. *Fix a triplet of integers $r, n \geq 2$ and $m \in \{1, .., n-1\}$. There exist positive constants $D(r, n, m) \in \mathbb{N}$, and $K_i = K_i(r, n, m) \in \mathbb{R}$, $i = 1, 2$, such that, for any $\xi > 0$, for any polynomial $Q \in \mathcal{P}(r, n)$ satisfying $\nabla Q(0) \neq 0$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla Q(0)$, there exists a semi-algebraic curve $\phi = (\phi_1(\eta), ..., \phi_m(\eta)) : [0, \xi] \longrightarrow \mathcal{T}(Q, \Gamma^m)$ having the following properties*

1. *For any fixed $\eta \in [0, \xi]$, the intersection $Im(\phi) \cap \mathcal{S}^m_\eta$ is a singleton;*

2. *The diagram of graph(ϕ) (see Definition A.1.2) is bounded by D;*

3. *There exists a closed interval $\mathcal{I}_\xi \subset [0, \xi]$ of length ξ / K_1 over which ϕ is real-analytic, with complex analyticity width ξ / K_1;*

4. *On the closed complex polydisk $(\mathcal{I}_\xi)_{\xi/K_1}$ of width ξ/K_1 around $\mathcal{I}_\xi$, one has the uniform Bernstein's inequality*

$$\max_{\eta \in (\mathcal{I}_\xi)_{\xi/K_1}} |\phi_j(\eta)| \leq K_2\, \xi \qquad \textit{for any } j \in \{1, ..., m\}.$$

Proof. As in Lemma 5.0.1, we consider the isomorphism $\Gamma^m \simeq \mathbb{R}^m$, and a global system of orthonormal coordinates $x = (x_1, ..., x_m)$. We proceed by steps. At the first step, we build a semi-algebraic function ϕ_1 associating to a sphere $\mathcal{S}^m_\eta$ of given radius $\eta > 0$ the minimal value attained by the coordinate x_1 on $\mathcal{S}^m_\eta$. At Step 2 we apply Yomdin's reparametrization to one of the algebraic components of ϕ_1 and we get a function with the suitable properties. Finally, at Step 3, we repeat the same construction for the other coordinates.

Step 1. For any $\xi > 0$, by Lemma 5.0.1 the set

$$\mathcal{T}_\xi(Q, \Gamma^m) := \mathcal{T}(Q, \Gamma^m) \cap \{x \in \mathbb{R}^m \,:\, ||x||_2^2 \leq \xi^2\}$$

is semi-algebraic and its diagram is bounded by a positive constant $\mathsf{d} = \mathsf{d}(r, n, m)$. By the Theorem of Tarski and Seidenberg A.1.1, the continuous function

$$f_1 := \mathcal{T}_\xi(Q, \Gamma^m) \longrightarrow \mathbb{R} \quad , \qquad x \longmapsto x_1$$

is semi-algebraic and its diagram is bounded by a quantity depending only on r, n, m. Infact,

$$\text{graph}(f_1) := \Pi_{\mathbb{R}^m \times \mathbb{R}}\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \mid u \in \mathcal{T}_\xi(Q, \Gamma^m), u = v\} \;.$$

Moreover, Lemma 5.0.1 assures that for any $0 < \eta_0 < \xi$ the set $\mathcal{T}_\xi(Q, \Gamma^m) \cap \mathcal{S}^m_{\eta_0}$ is compact, so that f_1 admits minimum on it. On the other hand, the function $g_1 : \mathcal{T}_\xi(Q, \Gamma^m) \longrightarrow \mathbb{R}$, $x \longmapsto ||x||_2$ is also semi-algebraic of diagram by a quantity depending only on r, n, m, since

$$\text{graph}(g_1) := \{(x, y) \in \mathbb{R}^m \times \mathbb{R} \mid x \in \mathcal{T}_\xi(Q, \Gamma^m)\,,\;\; ||x||_2^2 - y^2 = 0\} \;.$$

Then, by applying Proposition A.1.11, we have that the function

$$\phi_1 \,:\, [0, \xi] \longrightarrow \mathbb{R} \qquad \eta \longmapsto \inf_{x \in g_1^{-1}(\eta)} f_1(x) = \min_{x \in \mathcal{T}_\xi(Q, \Gamma^m) \cap \mathcal{S}^m_\eta} \{x_1\} \tag{5.0.4}$$

is semi-algebraic and we indicate by $d_1 = d_1(r, n, m)$ its diagram.

Step 2. Corollary A.1.1 ensures the existence of a number $N_1 = N_1(d_1)$ and of an open interval $\mathcal{I}^1 \subset [0, \xi]$ of length ξ / N_1 over which the restriction $\phi_1|_{\mathcal{I}^1}$ is algebraic. By Proposition A.1.6, $\phi_1|_{\mathcal{I}^1}$ is d_1-valent and has no more than d_1 zeros on its domain so that there exists an interval $\mathcal{J}^1 \subset \mathcal{I}^1$ of length $\xi/(N_1(d_1) \times (d_1 + 1))$ over which the restriction $\phi_1|_{\mathcal{J}^1}$ has definite sign. Without loss of generality we can assume $\phi_1(\eta) \geq 0$ for all $\eta \in \mathcal{J}^1$ (one considers $-\phi_1$ otherwise).

We denote by ξ_1 and ξ_2 the extremal points of the interval $\mathcal{J}^1$ and we rescale the domain by setting

$$\varphi_1 \,:\, [-1, 1] \longrightarrow \mathbb{R} \quad , \qquad \varphi_1(u) := \phi_1\left(\frac{u+1}{2}\xi_2 - \frac{u-1}{2}\xi_1\right) \;. \tag{5.0.5}$$

We also define the function

$$\widetilde{\varphi}_1(u) := \frac{\varphi_1(u)}{\xi} \quad , \qquad 0 \le \widetilde{\varphi}_1(u) \le 1 \, , \tag{5.0.6}$$

which satisfies the hypotheses of Theorem A.2.1. With the notations of Theorem A.2.1, we choose the value $\delta := \dfrac{1}{8\mathtt{Y}_1(d_1)}$ so that, once at most $\mathtt{Y}_1$ neighborhoods of length 2δ around the singularities of $\widetilde{\varphi}_1$ are eliminated from $[-1,1]$, the remaining set has a measure which is no less than $2 - \sum_{i=1}^{\mathtt{Y}_1} 2\times 1/(8\mathtt{Y}_1) = 7/4$. Moreover, the number of the partition intervals is bounded by the uniform quantity $\mathtt{Y}_2 \log_2\left(8\mathtt{Y}_1\right)$, so that there exists an interval $\Delta_1 \subset [-1,1]$ satisfying

$$|\Delta_1| = \frac{7}{4\,\mathtt{Y}_2 \log_2\left(8\mathtt{Y}_1\right)}$$

on which $\widetilde{\varphi}_1$ is real-analytic. Infact, by Proposition A.2.1, the complex singularities of $\widetilde{\varphi}_1$ are at distance no less than $3|\Delta_1|$ from the center c_1 of Δ_1. By Theorem A.2.1, Δ_1 can be affinely reparametrized by a function $\psi_1 : [-1,1] \longrightarrow \Delta_1$, which maps the complex disc $\mathcal{D}_3(0)$ into the complex disc $\mathcal{D}_\varrho(c_1)$ of radius $\varrho := \frac{3}{2}|\Delta_1|$. Hence, we can write

$$\begin{aligned} \max_{u\in\mathcal{D}_\varrho(c_1)} |\widetilde{\varphi}_1(u)| &\le \max_{u\in\mathcal{D}_\varrho(c_1)} |\widetilde{\varphi}_1(u) - \widetilde{\varphi}_1(0)| + |\widetilde{\varphi}_1(0)| \\ &= \max_{z\in\mathcal{D}_3(0)} |\widetilde{\varphi}_1\circ\psi_1(z) - \widetilde{\varphi}_1\circ\psi_1(0)| + |\widetilde{\varphi}_1(0)| \end{aligned} \tag{5.0.7}$$

so that, by Definition A.2.1 and Theorem A.2.1 and by the fact that $|\widetilde{\varphi}_1(u)| \le 1$ for any $u \in [-1,1]$, we obtain

$$\max_{u\in\mathcal{D}_\varrho(c_1)} |\widetilde{\varphi}_1(u)| \le 2 \, . \tag{5.0.8}$$

Scaling back to the original variables, by (5.0.5) the interval Δ_1 is mapped into an interval Δ_1^ξ of length $|\Delta_1^\xi| = |\Delta_1|\frac{\xi_2-\xi_1}{2} = |\Delta_1|\frac{\xi}{2N_1(d_1)\times(d_1+1)}$ and center $\widehat{c}_1$ and, in the same way, the radius rescales as $\varrho \rightsquigarrow \varrho\frac{\xi}{2N_1(d_1)\times(d_1+1)}$. Therefore, taking into account (5.0.6) and (5.0.8), there exists a uniform constant $\mathtt{M}_1 = \mathtt{M}_1(d_1)$ such that the following Bernstein's inequality is satisfied

$$\max_{z\in\mathcal{D}_{\xi/\mathtt{M}_1}(\widehat{c}_1)} |\phi_1(z)| \le 2\xi \, . \tag{5.0.9}$$

Step 3. Since Δ_1^ξ is compact, $\phi_1(\Delta_1^\xi)$ is also compact and the inverse image $\mathtt{U}_\xi^1(Q,\Gamma^m) := g_1^{-1}(\phi_1^{-1}(\phi_1(\Delta_1^\xi)))$ is closed. Moreover, since the diagrams of ϕ_1 and g_1 depend only on r,n,m, then by Propositions A.1.7-A.1.8-A.1.9 the diagram of $\mathtt{U}_\xi^1(Q,\Gamma^m)$ also depends only on r,n,m. Hence, for any fixed $\eta \in \Delta_1^\xi$ we have that the set $\mathcal{S}_\eta^m \cap$

$\mathtt{U}^1_\xi(Q, \Gamma^m)$, which contains the points of the Thalweg that have minimal coordinate x_1 on the sphere of radius η, is compact and semialgebraic with a bound on its diagram depending only on r, n, m. Hence, the coordinate x_2 admits a minimum on this set and we can repeat the same argument of Step 2 on the function

$$\phi_2 : \Delta_1^\xi \longrightarrow \mathbb{R} \qquad \eta \longmapsto \min_{x \in g_2^{-1}(\eta)} f_2(x) = \min_{x \in \mathcal{S}^m_\eta \cap \mathtt{U}^1_\xi(Q,\Gamma^m)} \{x_2\} \tag{5.0.10}$$

where we have set $f_2 : \mathcal{S}^m_{\eta_0} \cap \mathtt{U}^1_\xi(Q, \Gamma^m) \longrightarrow \mathbb{R}$, $x \longmapsto x_2$ and $g_2 : \mathcal{S}^m_{\eta_0} \cap \mathtt{U}^1_\xi(Q, \Gamma^m) \longrightarrow \mathbb{R}, x \longmapsto ||x||_2$. The curve $\phi := (\phi_1, \phi_2, ..., \phi_m)$ is constructed by iterating this procedure m times.

Points 1, 2, and 3 of the thesis follows easily from this construction. Point 4 is a consequence of estimate (5.0.9) applied to the complex polydisk of uniform width ξ/K_1 around the common uniform real interval of analyticity $\mathcal{I}_\xi$ of the functions $\phi_1, ..., \phi_m$. □

We are now ready to state the proof of Theorem 5.0.1.

Proof. (Theorem 5.0.1) We assume the setting of Lemma 5.0.2 with Q equal to the Taylor expansion $\mathtt{T}_0(h, r, n)$, and we proceed by steps. At Step 1, we show that there exists a component ϕ_i, $i \in \{1, \dots, m\}$, of the curve ϕ introduced in Lemma 5.0.2 whose first derivative admits a lower bound on a domain of uniform length. Then, at the second step, we use this fact to apply a quantitative inverse function Theorem and we reparametrize ϕ by the i-th coordinate. Steps 3 and 4 contain, respectively, the proofs of points 1-4 and of point 5 in the statement.

Step 1. We cut the uniform interval of analyticity $\mathcal{I}_\xi$ into three equal intervals and we denote by $\widehat{\mathcal{I}}_\xi$ the central one, whose length is $|\widehat{\mathcal{I}}_\xi| = |\mathcal{I}_\xi|/3$. We indicate by $\widehat{\xi}_1$ and $\widehat{\xi}_2$ the extreme points of $\widehat{\mathcal{I}}_\xi$. Since for any given $\eta \in \widehat{\mathcal{I}}_\xi$ by Lemma 5.0.2 we have $\eta^2 = \phi_1^2(\eta) + ... + \phi_m^2(\eta)$, there must be some component ϕ_i of the curve, with $i \in \{1, ..., m\}$, satisfying

$$|\phi_i(\widehat{\xi}_2) - \phi_i(\widehat{\xi}_1)| \ge \frac{|\widehat{\mathcal{I}}_\xi|}{m} = \frac{\xi}{3mK_1} \,. \tag{5.0.11}$$

At the same time, for some point $\widehat{\xi}_3 \in \widehat{\mathcal{I}}_\xi$ we have

$$|\phi_i(\widehat{\xi}_2) - \phi_i(\widehat{\xi}_1)| = |\phi_i'(\widehat{\xi}_3)||\widehat{\mathcal{I}}_\xi| = |\phi_i'(\widehat{\xi}_3)|\frac{\xi}{3K_1} \,. \tag{5.0.12}$$

On the one hand, relations (5.0.11) and (5.0.12) together imply

$$|\phi_i'(\widehat{\xi}_3)| \ge \frac{1}{m} \,. \tag{5.0.13}$$

On the other hand, for any $\eta \in [\widehat{\xi}_3 - |\widehat{\mathcal{I}}_\xi|, \widehat{\xi}_3 + |\widehat{\mathcal{I}}_\xi|] \subset \mathcal{I}_\xi$ one has the estimate

$$|\phi_i'(\eta) - \phi_i'(\widehat{\xi}_3)| \le \max_{\mathcal{I}_\xi} |\phi_i''||\eta - \widehat{\xi}_3| \tag{5.0.14}$$

which, thanks to the classic Cauchy estimate and to the Bernstein inequality of Lemma 5.0.2, implies

$$|\phi_i'(\eta) - \phi_i'(\widehat{\xi}_3)| \le \frac{2K_1^2}{\xi^2} \max_{(\mathcal{I}_\xi)_{\xi/K_1}} |\phi_i||\eta - \widehat{\xi}_3| \le \frac{2K_1^2 K_2}{\xi} |\eta - \widehat{\xi}_3| \, . \tag{5.0.15}$$

Hence, for any η in the interval $J_\xi := \left[\widehat{\xi}_3 - \frac{\xi}{4mK_1^2K_2}, \widehat{\xi}_3 + \frac{\xi}{4mK_1^2K_2}\right] \subset \mathcal{I}_\xi$ we have by (5.0.13) and (5.0.15) that

$$|\phi_i'(\eta)| \ge \Big| |\phi_i'(\widehat{\xi}_3)| - |\phi_i'(\eta) - \phi_i'(\widehat{\xi}_3)| \Big| \ge \frac{1}{m} - \frac{2K_1^2K_2}{\xi} \frac{\xi}{4mK_1^2K_2} = \frac{1}{2m} \, . \tag{5.0.16}$$

Step 2. By Lemma 5.0.2 and by the construction at Step 1 we can apply the quantitative local inversion Theorem B.0.2 for ϕ_i at any point $\eta \in J_\xi \subset \mathcal{I}_\xi$. By making use of the notations in Theorem B.0.2, we can set the uniform parameters

$$R := \frac{\xi}{K_1} \quad , \qquad |\phi_i'(\eta)| \ge \frac{1}{2m} \quad , \qquad \max_{(J_\xi)_{\xi/K_1}} |\phi_j''| \le 2\frac{K_1^2K_2}{\xi} \, . \tag{5.0.17}$$

Hence, ϕ_i is invertible in the complex closed polydisk $(J_\xi)_{R'/16}$ around the real interval J_ξ, where

$$R' := \frac{1}{2} \times \min\left\{ R, \frac{\min_{J_\xi} |\phi_i'|}{\max_{(J_\xi)_{\xi/(2K_1)}} |\phi_i''|}, \right\} = \frac{\xi}{8mK_1^2K_2} \, .$$

Since, by construction, ϕ_i is real-analytic in J_ξ, the continuity of the derivative ensures that $\phi_i(J_\xi)$ is an interval of $\mathbb{R}$. The inverse function is analytic in the complex polydisc of uniform width

$$R'' := \min_{J_\xi} |\phi_i'| \frac{R'}{8} \ge \frac{R'}{16m} = \frac{\xi}{128\, m^2 K_1^2 K_2} \tag{5.0.18}$$

around $\phi_i(J_\xi)$. Using (5.0.16), one has that its length is no less than

$$|\phi_i(J_\xi)| \ge \min_{J_\xi} |\phi_i'| \times |J_\xi| \ge \frac{1}{2m} \times \frac{\xi}{2mK_1^2K_2} \, . \tag{5.0.19}$$

Step 3. Point 1 of Theorem 5.0.1 follows by Point 1 of Lemma 5.0.2 and by the local inversion Theorem applied at Step 2. Points 2, 3, and 4 of Theorem 5.0.1 are also immediate consequences of the local inversion Theorem at Step 2.

Namely, by keeping in mind the notations at Point 4 of Theorem 5.0.1, the curve $\gamma := \phi \circ \phi_i^{-1}$ can be defined as

$$\gamma(t) := \begin{cases} x_i = t \\ x_j(t) = f_j(x_i) := \phi_j(\phi_i^{-1}(x_i)) \qquad \text{for all } j \in \{1, ..., m\} \, , \quad j \ne i \, . \end{cases} \tag{5.0.20}$$

The existence of an interval of analyticity with uniform length and complex width for γ is a consequence of (5.0.18) and (5.0.19) and the constant $\mathtt{K}$ in the statement can be taken equal to

$$\mathtt{K} := 128\, m^2 K_1^2 K_2 \,. \tag{5.0.21}$$

Indeed, for any $0 < \lambda \leq \xi$, $\mathtt{I}_\lambda$ can be chosen to be any interval of length $\lambda/\mathtt{K}$ contained in the interval $\phi_i(J_\lambda) \subset [-\lambda, \lambda]$ (see (5.0.19)). For later convenience, we also observe that the above discussion implies that

$$(\mathcal{I}_\lambda)_{\lambda/K_1} \supset (J_\lambda)_{\lambda/K_1} \supset \phi_i^{-1}(\mathtt{I}_\lambda)_{\lambda/\mathtt{K}} \quad , \qquad \forall \lambda \in (0, \xi] \,. \tag{5.0.22}$$

The fact that the diagram of graph(γ) depends only on r, n, m is an immediate consequence of (5.0.20), together with point 2 of Lemma 5.0.2 and with Propositions A.1.8-A.1.9.

Step 4. It remains to prove the Bernstein's inequality at Point 5 of the statement. By Lemma 5.0.2, for any $\lambda > 0$ we have

$$\max_{\eta \in (\mathcal{I}_\lambda)_{\lambda/K_1}} |\phi_j(\eta)| \leq K_2\, \lambda \tag{5.0.23}$$

for some uniform constant $K_2 = K_2(r, n, m)$ and for any $j \in \{1, ..., m\}$. By construction in (5.0.20), $f_j(x_1) = \phi_j \circ \phi_i^{-1}(x_i)$, and for any $\beta \in \mathbb{N} \cup \{0\}$ the classic Cauchy estimate implies

$$\max_{t \in \mathtt{I}_\lambda} |f_j^{(\beta)}(t)| \leq \beta!\, \mathtt{K}^\beta \frac{\max_{z \in (\mathtt{I}_\lambda)_{\lambda/\mathtt{K}}} |f_j(z)|}{\lambda^\beta} = \beta!\, \mathtt{K}^\beta \frac{\max_{z \in (\mathtt{I}_\lambda)_{\lambda/\mathtt{K}}} |\phi_j \circ \phi_i^{-1}(z)|}{\lambda^\beta} \,. \tag{5.0.24}$$

For any $0 < \lambda \leq \xi$, by (5.0.22) one has $\phi_i^{-1}((\mathtt{I}_\lambda)_{\lambda/\mathtt{K}}) \subset (\mathcal{I}_\lambda)_{\lambda/K_1}$. Taking this into account, (5.0.24) and (5.0.23) yield

$$\max_{t \in \mathtt{I}_\lambda} |f_j^{(\beta)}(t)| \leq \beta!\, \mathtt{K}^\beta \frac{\max_{\eta \in (\mathcal{I}_\lambda)_{\lambda/K_1}} |\phi_j(\eta)|}{\lambda^\beta} \leq \beta!\, \mathtt{K}^\beta \frac{K_2 \lambda}{\lambda^\beta} = \beta!\, \mathtt{K}^\beta \frac{K_2}{\lambda^{\beta-1}} \,. \tag{5.0.25}$$

The thesis at Point 5 in the statement follows by setting $\mathtt{M} = \beta! \times K_2 \times \mathtt{K}^\beta$.

□

Chapter 6

s-vanishing polynomials

We take into account the results and the notations of the previous section, in particular Theorem 5.0.1.

6.1 Heuristics and Definitions

The goal of the first part of this paragraph is to provide the reader with a heuristic justification for introducing the special class of s-vanishing polynomials in the study of the genericity of steepness. A rigorous description of the rôle played by these polynomials will be given in the next paragraphs and sections.

For any fixed integer $n \geq 2$, we consider the euclidean space $\mathbb{R}^n$ and we endow any of its linear subspaces with the induced metric. For any pair of positive integers $1 \leq m \leq n-1$ and $r \geq 2$, for any given function h of class C^r near the origin satisfying $\nabla h(0) \neq 0$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla h(0)$, by Def. 11.2.2, the set $\mathcal{T}(\mathtt{T}_0(h,r,n),\Gamma^m)$ is the locus of minima of $||\pi_{\Gamma^m}\nabla\mathtt{T}_0(h,r,n)||_2$ on the spheres $\mathcal{S}^m_\eta(0) \subset \Gamma^m$, with $\eta > 0$.

In Theorem 5.0.1 we have proved the existence of a minimal semi-algebraic arc γ (see (5.0.20)) of diagram $\mathtt{d}(r,n,m)$ parametrized by one coordinate and whose image is contained in the thalweg $\mathcal{T}(\mathtt{T}_0(h,r,n),\Gamma^m)$. Due to Proposition A.1.4 - γ is piecewise algebraic, with a maximal number of algebraic components depending only on its diagram $\mathtt{d}(r,n,m)$. With the exception of a finite set of complex points, any algebraic function admits locally a holomorphic extension, and the number of its singularities is bounded by a quantity depending only on its diagram (see appendix A.2, or [16] for more details). Therefore, $\gamma(t)$ is real-analytic with the exception of a finite number of points whose cardinality is bounded uniformly by a quantity depending solely on $\mathtt{d}(r,n,m)$. In particular, for any $\lambda > 0$, this ensures the existence of an interval $\mathtt{I}_\lambda \subset [-\lambda,\lambda]$ of uniform length $\lambda/\mathtt{K}(r,m,n)$, where $\mathtt{K} = \mathtt{K}(r,m,n)$ is a suitable constant, over which $\gamma(t)$ is real-analytic with complex analyticity width $\lambda/\mathtt{K}$.

By the above reasonings, for sufficiently small $\lambda > 0$ the interval $(-3\lambda, 3\lambda)$ contains no singularities of $\gamma(t)$. In particular, $\gamma(t)$ is real analytic in $\mathtt{L}_\lambda := (\lambda, 2\lambda)$, with complex analyticity width λ, and the same holds also for $|| \, \pi_{\Gamma^m} \nabla \mathtt{T}_0(h,r,n)\big|_{\gamma(t)} \, ||_2{}^2$ in that interval. Hence, if the function $|| \, \pi_{\Gamma^m} \nabla \mathtt{T}_0(h,r,n)\big|_{\gamma(t)} \, ||_2{}^2$ has a zero of infinite order at some point $t^\star \in \mathtt{L}_\lambda$, then it is identically null in $\mathtt{L}_\lambda$. Then, by Definition 11.2.1, and by the minimality of γ, this implies that the polynomial $\mathtt{T}_0(h,r,n)$ cannot satisfy the steepness property at the origin on the subspace Γ^m.

We claim that a kind of converse result - involving $\mathtt{I}_\lambda$ instead of $\mathtt{L}_\lambda$ - is also true: if $\mathtt{T}_0(h,r,n)$ is non-steep at the origin on the subspace Γ^m, then $|| \, \pi_{\Gamma^m} \nabla \mathtt{T}_0(h,r,n)\big|_{\gamma(t)} \, ||_2{}^2$ must have a zero of infinite order in $\mathtt{I}_\lambda$. This observation is fundamental in order to prove Theorem A. Actually, the necessity of a zero of infinite order has been proved for a real-analytic function in [98] via the curve-selection Lemma; however, in the polynomial setting considered here, we have a much stronger quantitative result.

Motivated by this heuristic argument, we are interested in studying the properties of those real polynomials of $m \geq 1$ variables whose gradient has a zero of sufficiently high order on some curve γ parametrized by one coordinate. In a first moment, we do not consider the fact that these polynomials are the restrictions to a m-dimensional subspace Γ^m of polynomials defined in $\mathbb{R}^n$, with $n > m$. This will be taken into account in section 7. Therefore, we give the following definitions:

Definition 6.1.1. We indicate by Θ_m the set of curves $\gamma(t)$ with values in $\mathbb{R}^m$ such that

1. $\gamma(t)$ is real-analytic in a neighborhood U_γ of the origin, and $\gamma(0) = 0$;

2. for some $k \in \{1, ..., m\}$, and for all $t \in U$

$$\gamma(t) := \begin{cases} x_k(t) = t \\ x_j(t) = f_j(t) \quad \forall j \in \{1, ..., m\} \, , \quad j \neq k \, . \end{cases}$$

For fixed $i \in \{1, \dots, m\}$, we denote by Θ_m^i the subset of curves in Θ_m that are parametrized by the i-th coordinate.

Remark 6.1.1. We are asking the arc γ to be analytic at the origin, but the minimal arc obtained in Theorem 5.0.1 did not necessarily have this property (the origin was not included, in general, in the uniform interval of analyticity $\mathtt{I}_\lambda \subset [-\lambda, \lambda]$). As it has already been discussed in the introduction, this is an issue that comes from the use of analytic reparametrizations of semi-algebraic sets. We will deal with this apparent difficulty in section 7.

Remark 6.1.2. For the moment, we do not make any assumption on the sizes of the neighborhoods of analyticity U_γ of the arcs in Θ_m. Hence, the results of this section do not require any uniform lower bound on $|U_\gamma|$ as in Theorem 5.0.1. Nevertheless, the existence of a uniform lower bound will prove to be necessary in order to to demonstrate the results of section 7.

Remark 6.1.3. The definition of the set Θ_m is coordinate-dependent. However, as we have showed in Chapter 5 (see Theorem 5.0.1), for any function $h \in C^r(B^n(0,2\delta))$ verifying $\nabla h(0) \neq 0$, and for any euclidean suspace Γ^m orthogonal to $\nabla h(0)$, there exists a minimal arc $\gamma \in \Theta_m$ with uniform characteristics whose image is contained in the thalweg $\mathcal{T}(\mathrm{T}_0(h,r,n),\Gamma^m)$.

Definition 6.1.2. For any pair of integers $s \geq 1$ and $i \in \{1,\dots,m\}$, we indicate by $\vartheta_m(s)$ (resp. $\vartheta^i_m(s)$) the subset of $(\mathcal{P}^\star(s,1))^m = \mathcal{P}^\star(s,1)\times\cdots\times\mathcal{P}^\star(s,1)$ containing the truncations at order s of the Taylor expansions at the origin of all curves in Θ_m (resp. in Θ^i_m). The elements of $\vartheta_m(s)$ will henceforth be referred to as s-truncations.

Remark 6.1.4. Clearly, $\vartheta_m(s)$ (resp. $\vartheta^i_m(s)$) is isomorphic to the set of s-jets of curves in Θ_m (resp. Θ^i_m). Moreover, the set $\vartheta^i_m(s)$ is isomorphic to $\mathbb{R}^{(m-1)\times s}$, since for any curve $\gamma \in \Theta^i_m$ its s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^i_m(s)$ is determined by the first s Taylor coefficients at the origin of the functions f_j, with $j \in \{1,...,m\}$, $j \neq i$.

Remark 6.1.5. With the definitions above, one has the following decompositions:

$$\Theta_m = \bigcup_{i=1}^m \Theta^i_m \quad , \qquad \vartheta_m(s) = \bigcup_{i=1}^m \vartheta^i_m(s)\ . \tag{6.1.1}$$

Definition 6.1.3. Fix three integers $r \geq 2$, $m \geq 1$, and $1 \leq s \leq r-1$. A polynomial $P \in \mathcal{P}(r,m)$ is said to be *s-vanishing* if there exists an arc $\gamma \in \Theta_m$ such that on its s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta_m(s)$ the gradient of P has a zero of order s at the origin, namely

$$\frac{d^\alpha}{dt^\alpha}\left(\left.\frac{\partial P}{\partial x_\ell}\right|_{\mathcal{J}_{s,\gamma}(t)}\right)_{t=0} = 0\ ,\quad \forall\,\ell \in \{1,...,m\}\ ,\ \forall\,\alpha \in \{0,...,s\}\ . \tag{6.1.2}$$

The set of s-vanishing polynomials in $\mathcal{P}(r,m)$ is denoted by $\sigma(r,s,m)$.

In paragraph 6.2, we shall investigate the properties of the set $\sigma(r,s,m) \subset \mathcal{P}(r,m)$ of s-vanishing polynomials: we shall prove that

1. for any given value of $r \geq 2$, $m \geq 1$, $1 \leq s \leq r-1$, it is the semi-algebraic projection onto $\mathcal{P}(r,m)$ of an algebraic set $Z(r,s,m)$ of $\mathcal{P}(r,m)\times\vartheta_m(s)$ whose ideal can be explicitly computed ;

2. it has positive codimension.

Secondly, in paragraph 6.3, we shall show that any polynomial P belonging to the complementary of the closure of $\sigma(r,s,m)$ in $\mathcal{P}(r,m)$ satisfies a "stable" lower estimate on its gradient. As we shall see, "stable" means that the estimate holds uniformly true for any polynomial belonging to a neighborhood of P.

Finally, in section 7, we shall prove that a polynomial $Q \in \mathcal{P}(r,n)$ satisfying $\nabla Q(0) \neq 0$ is steep around the origin iff there exists $1 \leq s \leq r-1$ such that, for all $m \in \{1,...,n-1\}$, the restriction of Q to any m-dimensional linear subspace Γ^m perpendicular to $\nabla Q(0)$ is contained in the complementary of closure($\sigma(r,s,m)$) in $\mathcal{P}(r,m)$.

6.2 Algebraic properties

We assume the notations of the previous paragraph, and we consider a triplet of integers $r \geq 2$, $m \geq 1$, $1 \leq s \leq r-1$. We work in the euclidean space $\mathbb{R}^m$ equipped with coordinates $(x_1, ..., x_m)$, and we consider a polynomial $P = P(x) \in \mathcal{P}(r,m)$ satisfying the s-vanishing condition on the s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta_m(s)$ of some curve $\gamma(t) := (x_1(t), ..., x_m(t)) \in \Theta_m$. Unless explicitly specified, we will henceforth work in the case in which γ is parametrized by the first coordinate, as the generalization to other cases is immediate. Hence, $\gamma(t) := (t, x_2(t), \dots, x_m(t)) \in \Theta_m^1$.

6.2.1 Case $m = 1$

We observe that, for $m = 1$, we have the following simple result:

Lemma 6.2.1. *For $m = 1$, a polynomial $P(x) = \sum_{\substack{\mu\in\mathbb{N}\\ 1\leq\mu\leq r}} p_\mu x^\mu$ of one real variable belongs to the set $\sigma(r,s,1) \subset \mathcal{P}(r,1)$ if and only if*

$$p_\mu = 0 \qquad \forall \mu \in \mathbb{N} \text{ such that } 1 \leq \mu \leq s+1 \ .$$

Moreover, $\sigma(r,s,1)$ is closed and its codimension in $\mathcal{P}(r,1)$ is equal to $s+1$.

Proof. For $m = 1$, the set Θ_1 is the singleton containing the line $\gamma(t) := x(t) = t$.

By Definition 6.1.3, it is clear that a polynomial verifying the hypotheses in the statement satisfies also the s-vanishing condition. Conversely, again by Definition 6.1.3, it is plain to check that the s-vanishing condition for $m = 1$ imposes that the coefficients of the studied polynomial must be null up to order $s+1$. The closure of $\sigma(r,s,1)$ is due to continuity, whereas codim $\sigma(r,s,1) = s+1$ as such a set is determined by $s+1$ independent equations in $\mathcal{P}(r,1)$. □

6.2.2 Notations (case $n \geq 3$, $2 \leq m \leq n-1$)

Up to the end of this paragraph, we will restrict to the case $n \geq 3$, $2 \leq m \leq n-1$. The goal is to introduce useful notations in order to study the properties of s-vanishing polynomials in $\mathcal{P}(r,m)$.

Using standard notations, we set $\mu := (\mu_1, ..., \mu_m) \in \mathbb{N}^m$, $|\mu| := ||\mu||_1$, and for any $P \in \mathcal{P}(r,m)$ we write

$$P(x) := \sum_{\substack{\mu\in\mathbb{N}^m\\ 1\leq|\mu|\leq r}} p_{(\mu_1,...,\mu_m)} x_1^{\mu_1} ... x_m^{\mu_m} =: \sum_{\substack{\mu\in\mathbb{N}^m\\ 1\leq|\mu|\leq r}} p_\mu x^\mu \ , \tag{6.2.1}$$

where we have taken into account the fact that P has no constant term by the definition of $\mathcal{P}(r,m)$.

We also consider a curve $\gamma \in \Theta^1_m$ and - for $j = 2, ..., m$ - we develop its components f_j at the origin, and we write

$$x_j(t) = f_j(t) =: \sum_{i=1}^{+\infty} a_{ji} t^i \ , \tag{6.2.2}$$

where we have taken into account the fact that $\gamma(0) = 0$ by Definition 6.1.1. Thus, the s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ of the curve γ is identified by the $(m-1)s$ real coefficients $(a_{21}, ..., a_{2s}, ..., a_{m1}, ..., a_{ms})$ of the truncated expansion, namely

$$\mathcal{J}_{s,\gamma} = \mathcal{J}_{s,\gamma}(t) = \left(t, \sum_{i=1}^{s} a_{2i} t^i \ , \ldots \ , \sum_{i=1}^{s} a_{mi} t^i \right) . \tag{6.2.3}$$

In the rest of this paragraph, we will try to find an explicit expression for the s-vanishing condition in terms of the coefficients of P and $\mathcal{J}_{s,\gamma}$. We first observe that the s-vanishing condition (6.1.2) for $\alpha = 0$ implies

$$p_\mu = 0 \qquad \text{for all } \mu \in \mathbb{N}^m \text{ such that } |\mu| = 1 \ . \tag{6.2.4}$$

Thus, without any loss of generality, in (6.2.1) we can only consider the multi-indices $\mu \in \mathbb{N}^m$ that satisfy $2 \le |\mu| \le r$. Moreover, for $\ell = 1, ..., m$, the ℓ-th component of the gradient of P reads

$$\frac{\partial P}{\partial x_\ell} := \sum_{\substack{\mu \in \mathbb{N}^m \\ 2 \le |\mu| \le r}} \mu_\ell \, p_\mu x^{\widetilde{\mu}(\ell)} \ , \ \ \widetilde{\mu}_j(\ell) := \mu_j - \delta_{j\ell} \ , \ \ j = 1, ..., m \ , \ \ |\mu| = |\widetilde{\mu}(\ell)| + 1 \ , \tag{6.2.5}$$

where $\delta_{j\ell}$ is the Kronecker symbol. At this point, we indicate by

$$\Phi^1 \ : \ \mathcal{P}(r, m) \times \vartheta^1_m(s) \longrightarrow \mathbb{R}^M \times \mathbb{R}^{(m-1)s} \ , \ \ M := \dim \mathcal{P}(r, m) \tag{6.2.6}$$

the trivial chart associating $(P, \mathcal{J}_{s,\gamma}) \longmapsto (p_\mu, a_{21}, ..., a_{2s}, ..., a_{m1}, ..., a_{ms})$ and we define the functions $q^1_{\ell\alpha} \ : \ \mathbb{R}^M \times \mathbb{R}^{(m-1)s} \longrightarrow \mathbb{R} \ , \ \ \ell \in \{1, ..., m\} \ , \ \ \alpha \in \{0, ..., s\}$ in the following way:

$$\begin{aligned}
&\text{For } \alpha = 0 \qquad q^1_{\ell 0} \circ \Phi^1(P, \mathcal{J}_{s,\gamma}) := \left(\frac{\partial P}{\partial x_\ell}\Big|_{\mathcal{J}_{s,\gamma}(t)} \right)_{t=0} = p_{(0,...,0,1,0,...,0)} \\
&\text{For } \alpha \in \{1, ..., s\} \qquad q^1_{\ell\alpha} \circ \Phi^1(P, \mathcal{J}_{s,\gamma}) := \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial P}{\partial x_\ell}\Big|_{\mathcal{J}_{s,\gamma}(t)} \right)_{t=0} \\
&= \frac{d^\alpha}{dt^\alpha} \left[\sum_{\substack{\mu \in \mathbb{N}^m \\ 2 \le |\mu| \le r}} \mu_\ell \, p_\mu t^{\widetilde{\mu}_1(\ell)} \left(\sum_{k=1}^{s} a_{2k} t^k \right)^{\widetilde{\mu}_2(\ell)} \cdots \left(\sum_{j=1}^{s} a_{mj} t^j \right)^{\widetilde{\mu}_m(\ell)} \right]_{t=0}
\end{aligned} \tag{6.2.7}$$

where the "1" fills the ℓ-th slot in the multi-index at the rightest member of the first line and where the last line is obtained by injecting (6.2.3) into expression (6.2.5).

Remark 6.2.1. In a similar way, when γ is parametrized by the i-th coordinate, with $i \neq 1$, one can denote by $\Phi^i : \mathcal{P}(r,m) \times \vartheta^i_m(s) \longrightarrow \mathbb{R}^M \times \mathbb{R}^{(m-1)s}$ the chart associating $(P, \mathcal{J}_{s,\gamma}) \mapsto (p_\mu, a_{11}, ..., a_{1s}, ..., a_{(i-1)1}, ..., a_{(i-1)s}, a_{(i+1)1}, ..., a_{(i+1)s}, ..., a_{m1}, ..., a_{ms})$, and introduce the maps $q^i_{\ell\alpha} \circ \Phi^i(P, \mathcal{J}_{s,\gamma})$, $\ell \in \{1, \dots, m\}$, $\alpha \in \{0, \dots, s\}$ exchanging the rôle of the first coordinate with that of the i-th coordinate in (6.2.7).

Remark 6.2.2. Comparing expressions (6.2.7) with Definition 6.1.3 and expression (6.2.4), and taking Remark 6.2.1 into account, it is easy to see that any given polynomial $P \in \mathcal{P}(r,m)$ satisfies the s-vanishing condition on some truncation $\mathcal{J}_{s,\gamma} \in \vartheta_m(s)$ if and only if

$$q^i_{\ell\alpha} \circ \Phi^i(P, \mathcal{J}_{s,\gamma}) = 0\,, \quad \text{for some } i \in \{2, \dots, m\}\,, \text{ for all } \ell \in \{1, ..., m\}\,,\ \alpha \in \{0, ..., s\}\,. \tag{6.2.8}$$

By the above discussion, we see that the set of s-vanishing polynomials in $\mathcal{P}(r,m)$ is given by

$$\sigma(r,s,m) = \bigcup_{i=1}^{m} \sigma^i(r,s,m)\,, \tag{6.2.9}$$

where we have introduced the sets

$$\sigma^i(r,s,m) := \Pi_{\mathcal{P}(r,m)} Z^i(r,s,m) \tag{6.2.10}$$

and

$$\begin{aligned} Z^i(r,s,m) := \{(P, \mathcal{J}_{s,\gamma}) \in \mathcal{P}(r,m) \times \vartheta^i_m(s)\} | (P, \mathcal{J}_{s,\gamma}) \text{ satisfies} \\ q^i_{\ell\alpha} \circ \Phi^i(P, \mathcal{J}_{s,\gamma}) = 0 \text{ for all } \ell \in \{1, ..., m\}, \alpha \in \{0, ..., s\}\}\,. \end{aligned} \tag{6.2.11}$$

We also set

$$Z(r,s,m) := \bigcup_{i=1}^{m} Z^i(r,s,m)\,. \tag{6.2.12}$$

It turns out that the ideal of $Z(r,s,m)$ can be explicitly computed for any given value of the integers $r \geq 2$, $1 \leq s \leq r-1$, $m \geq 2$ [1], i.e., one can find explicit expressions for the quantities $q^i_{\ell\alpha} \circ \Phi^i(P, \mathcal{J}_{s,\gamma})$, for any value of $\ell \in \{1, ..., m\}$, $\alpha \in \{0, ..., s\}$, and $i \in \{1, ..., m\}$.

Before stating this result, for any given value of $i \in \{1, \dots, m\}$ we will introduce new global charts for $\mathcal{P}(r,m) \times \vartheta^i_m(s)$ which - though unessential for the validity of our results - yield nicer expressions for the equations $q^i_{\ell\alpha} \circ \Phi^i(P, \mathcal{J}_{s,\gamma}) = 0$ than the standard chart Φ^i. As it will be shown in the next paragraph, the variables a_{j1}, $j \neq i$, associated to the linear terms of the s-truncation $\mathcal{J}_{s,\gamma}$ can be incorporated in the coordinates of the polynomial P. This simplifies the calculations and yields more readable formulas.

[1] The case $m = 1$ is easier, see Lemma 6.2.1.

6.2.3 A useful chart for $\mathcal{P}(r,m) \times \vartheta_m(s)$ (case $n \geq 3$, $2 \leq m \leq n-1$)

Here too, we restrict to the case $n \geq 3$, $2 \leq m \leq n-1$.

Once again, we only consider the case in which γ is parametrized by the coordinate $i = 1$, the other cases being trivial generalizations. Some of the quantities introduced in the sequel should be labeled with an index 1, as their definition depends in an obvious way from the choice of the parametrizing coordinate. However, in order not to burden notations, we drop, when possible, the reference to the fact that we are considering the case $i = 1$.

In order to define a new chart for $\mathcal{P}(r,m) \times \vartheta^1_m(s)$, we start by observing that, if we denote by $A_1, ..., A_m$ the canonical basis associated to the coordinates $x_1, ..., x_m$ in $\mathbb{R}^m$, for any fixed vector $\mathtt{b} := (b_{21}, ..., b_{m1}) \in \mathbb{R}^{m-1}$, we can define the new parametric basis

$$v_{\mathtt{b}} := A_1 + b_{21}A_2 + b_{31}A_3 + ... + b_{m1}A_m \ , \ \ u_2 := A_2 \ , \ \ ... \ , \ \ u_m := A_m \qquad (6.2.13)$$

associated to the parametric change of variables

$$\mathcal{L}_{\mathtt{b}} : \mathbb{R}^m \longrightarrow \mathbb{R}^m \ , \quad (x_1, ..., x_m) \longmapsto \mathtt{y}(\mathtt{b}) := (\mathtt{y}_1, \mathtt{y}_2(\mathtt{b}), ..., \mathtt{y}_m(\mathtt{b})), \qquad (6.2.14)$$

where

$$\mathtt{y}_1 := x_1 \quad , \qquad \mathtt{y}_2 = \mathtt{y}_2(\mathtt{b}) := x_2 - b_{21}\, x_1 \quad \dots \quad \mathtt{y}_m = \mathtt{y}_m(\mathtt{b}) := x_m - b_{m1}\, x_1 \ . \qquad (6.2.15)$$

Obviously, for any fixed $\mathtt{b} \in \mathbb{R}^{m-1}$, the change of coordinates (6.2.15) in $\mathbb{R}^m$ induces a change of coordinates also in $\mathcal{P}(r,m)$. Infact, the pull-back of the polynomial $P(x)$ is indicated by

$$\mathtt{P}_{\mathtt{b}}(\mathtt{y}(\mathtt{b})) := P \circ \mathcal{L}^{-1}_{\mathtt{b}}(\mathtt{y}(\mathtt{b})) = \sum_{\substack{\mu \in \mathbb{N}^m \\ 2 \leq |\mu| \leq r}} p_\mu (x(\mathtt{y}(\mathtt{b})))^\mu =: \sum_{\substack{\mu \in \mathbb{N}^m \\ 2 \leq |\mu| \leq r}} \mathtt{p}_\mu(p_\mu, \mathtt{b}) \mathtt{y}^\mu(\mathtt{b}) \ , \qquad (6.2.16)$$

where the new coefficients $\mathtt{p}_\mu = \mathtt{p}_\mu(p_\mu, b_{21}, ..., b_{m1})$ are polynomial functions of coefficients p_μ and on the parameters $\mathtt{b} \in \mathbb{R}^{m-1}$. For any given $\mathtt{b} \in \mathbb{R}^{m-1}$, there is a $1-1$ correspondence between the quantities p_μ and $\mathtt{p}_\mu$, as they represent the coordinates of the same polynomial written in different bases. Moreover, by (6.2.2), in the new variables (6.2.15) the components of the push-forward $\mathcal{L}_{\mathtt{b}} \circ \mathcal{J}_{s,\gamma} = \mathcal{L}_{\mathtt{b}} \circ \mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ of any s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ read

$$\mathtt{y}_1(t) = t \quad , \qquad \mathtt{y}_j(t) := (a_{j1} - b_{j1})t + \sum_{i=2}^{s} a_{ji} t^i \quad , \qquad j \in \{2, ..., m\} \ . \qquad (6.2.17)$$

By looking at expression (6.2.17), we see that, for any $s \geq 2$ and for any given s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$, it is possible to find a parametric change of coordinates $\mathcal{L}_{\mathtt{b}}$ in $\mathbb{R}^m$ such that the image $\mathcal{L}_{\mathtt{b}} \circ \mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ has no linear terms except for the parametrizing component: it suffices to choose $\mathtt{b} = \mathtt{a} := (a_{21}, ..., a_{m1})$, with $(a_{21}, ..., a_{m1})$ the coefficients of the linear terms of $\mathcal{J}_{s,\gamma}$. Taking (6.2.16) into account, this also entails that, for any given truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$, there exists an associated set of coordinates $\mathtt{p}_\mu = \mathtt{p}_\mu(p_\mu, \mathtt{a})$, with $\mathtt{a} := (a_{21}, ..., a_{m1})$, in the space of polynomials $\mathcal{P}(r,m)$. Hence, we can parametrize the coordinates of any polynomial P through the linear coefficients of any truncation $\mathcal{J}_{s,\gamma}$.

Namely, we firstly observe that $\vartheta^1_m(1)$ is isomorphic to

$$\vartheta^1_m(1) \simeq \{\mathtt{a} := (a_{21}, \dots, a_{m1}) \in \mathbb{R}^{m-1}\} \ . \tag{6.2.18}$$

Also, indicating by $\vartheta^1_m(s,2)$ the subset of s-truncations having null linear terms, we have that

$$\vartheta^1_m(s,2) \simeq \{(a_{22}, \dots, a_{2s}, \dots, a_{m2}, \dots, a_{ms}) \in \mathbb{R}^{(m-1)(s-1)}\} \ . \tag{6.2.19}$$

Clearly, one has

$$\vartheta^1_m(s) = \vartheta^1_m(1) \times \vartheta^1_m(s,2) \ . \tag{6.2.20}$$

At this point, with the notation in (6.2.16), we define the invertible transformation

$$\mathscr{F}^1 : \mathcal{P}(r,m) \times \mathbb{R}^{m-1} \longrightarrow \mathbb{R}^M \times \mathbb{R}^{m-1} \quad , \qquad M := \dim \mathcal{P}(r,m) \tag{6.2.21}$$

associating

$$\left(P = \sum_{\substack{\mu \in \mathbb{N}^m \\ 1 \le |\mu| \le r}} p_\mu x^\mu, \mathtt{a} \right) \longmapsto \left(\mathtt{p}_{\mu'} \left((p_\mu)_{\substack{\mu \in \mathbb{N}^m \\ 1 \le |\mu| \le r}}, \mathtt{a} \right)_{\substack{\mu' \in \mathbb{N}^m \\ 1 \le |\mu'| \le r}}, \mathtt{a} \right) \tag{6.2.22}$$

and we indicate its image by

$$W^1(r,m) := \mathscr{F}^1(\mathcal{P}(r,m) \times \mathbb{R}^{m-1}) \ . \tag{6.2.23}$$

In other words, $W^1(r,m)$ is constructed by attaching to any point $\mathtt{a} \in \mathbb{R}^{m-1} \simeq \vartheta^1_m(1)$ the fiber of all polynomials in $\mathcal{P}(r,m)$ expressed in the variables (6.2.15) associated to the value $\mathtt{a}$.

Furthermore, setting

$$\mathtt{J}_{s,\gamma,\mathtt{a}} := \mathcal{L}_\mathtt{a} \circ \mathcal{J}_{s,\gamma} \tag{6.2.24}$$

we have $\mathtt{J}_{s,\gamma,\mathtt{a}} \in \vartheta^1_m(s,2)$ by construction because in the adapted variables - as we had shown in (6.2.17) by setting $\mathtt{b} = \mathtt{a}$ - with the exception of the parametrizing component, any truncation starts at order two. Taking the notation in (6.2.16) into account, we can also define the chart

$$\Upsilon^1 : \mathcal{P}(r,m) \times \vartheta^1_m(s) \longrightarrow W^1(r,m) \times \mathbb{R}^{(m-1)(s-1)} \tag{6.2.25}$$

associating

$$(P, \mathcal{J}_{s,\gamma}) \longmapsto \left(\mathtt{p}_{\mu'} \left((p_\mu)_{\substack{\mu \in \mathbb{N}^m \\ 1 \le |\mu| \le r}}, \mathtt{a} \right)_{\substack{\mu' \in \mathbb{N}^m \\ 1 \le |\mu'| \le r}}, \ \mathtt{a}, a_{22}, a_{23}, ..., a_{ms} \right) \ . \tag{6.2.26}$$

Remark 6.2.3. For further convenience, we also denote by

$$\mathfrak{U}^1 : \mathcal{P}(r,m) \times \vartheta^1_m(1) \longrightarrow W^1(r,m) \tag{6.2.27}$$

the restriction of Υ^1 to $\mathcal{P}(r,m)\times\vartheta^1_m(1)$. It is plain to check by formulas (6.2.16), (6.2.22), and (6.2.26) that $\mathfrak{U}^1$ is polynomial, invertible and that its inverse is a polynomial map.

Remark 6.2.4. The generalization of the arguments above to the case in which the curve $\gamma \in \Theta_m$ is parametrized by the i-th coordinate, with $i \in \{2, \dots, m\}$, is immediate. In particular, one can define functions $\mathscr{F}^i$, Υ^i, $\mathfrak{U}^i$, together with sets $W^i(r,m)$.

Remark 6.2.5. With slight abuse of notation, in the rest of this work we will often write $(\mathtt{P}_\mathtt{a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})$ and $(\mathtt{P}_\mathtt{a}, \mathtt{a})$ to indicate the points of $\Upsilon^1(P, \mathcal{J}_{s,\gamma})$ and $W^1(r,m)$ respectively.

For further convenience, we also observe that

Lemma 6.2.2. *Any polynomial $P \in \mathcal{P}(r,m)$ satisfies the s-vanishing condition*

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial P(x)}{\partial x_\ell}\bigg|_{\mathcal{J}_{s,\gamma}(t)} \right)_{t=0} = 0 \quad \forall \alpha \in \{0, ..., s\}\ ,\ \ \forall \ell \in \{1, ..., m\} \tag{6.2.28}$$

on the s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ of some curve $\gamma \in \Theta^1_m$, if and only if it satisfies

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial \mathtt{P}_\mathtt{a}(\mathtt{y})}{\partial \mathtt{y}_\ell}\bigg|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right)_{t=0} = 0 \quad \forall \alpha \in \{0, ..., s\}\ ,\ \ \forall \ell \in \{1, ..., m\}$$

in the adapted coordinates associated to the linear terms of $\mathcal{J}_{s,\gamma}$.

Proof. By (6.2.15) one has

$$\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}) := \begin{cases} x_1 = \mathtt{y}_1 \\ x_2 = a_{21}\,\mathtt{y}_1 + \mathtt{y}_2 \\ \dots \\ x_m = a_{m1}\,\mathtt{y}_1 + \mathtt{y}_m \end{cases} , \quad \mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{21} & 1 & 0 & \dots & 0 \\ a_{31} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} . \tag{6.2.29}$$

Indicating with $(\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}))_{k,\ell}$ the (k,ℓ)-th entry of the Jacobian of the inverse transformation $\mathcal{L}^{-1}_\mathtt{a}$ in (6.2.29), by the Leibniz formula one has

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}\left(\frac{\partial \mathtt{P}_\mathtt{a}(\mathtt{y})}{\partial \mathtt{y}_\ell}\bigg|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right)_{t=0} &:= \frac{d^\alpha}{dt^\alpha}\left(\frac{\partial (P\circ\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}))}{\partial \mathtt{y}_\ell}\bigg|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right)_{t=0} \\ =&\frac{d^\alpha}{dt^\alpha}\left[\sum_{k=1}^m \frac{\partial P}{\partial x_k}\circ\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y})\bigg|_{\mathcal{L}_\mathtt{a}\circ\mathcal{J}_{s,\gamma}(t)} \times (\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}))_{k,\ell}\bigg|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right]_{t=0} \\ =&\frac{d^\alpha}{dt^\alpha}\left[\sum_{k=1}^m \frac{\partial P(x)}{\partial x_k}\bigg|_{\mathcal{J}_{s,\gamma}(t)} \times (\mathcal{L}^{-1}_\mathtt{a}(\mathtt{y}))_{k,\ell}\bigg|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right]_{t=0} . \end{aligned} \tag{6.2.30}$$

Since the entries of the matrix $\mathcal{L}_{\mathtt{a}}^{-1}$ in (6.2.29) are constant, one has

$$(\mathcal{L}_{\mathtt{a}}^{-1}(\mathtt{y}))_{k,\ell}\Big|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} = (\mathcal{L}_{\mathtt{a}}^{-1}(\mathtt{y}))_{k,\ell} \, ,$$

so that, finally, (6.2.30) reads

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial \mathtt{P}_{\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell}\Big|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)}\right)_{t=0} = \sum_{k=1}^{m} \frac{d^\alpha}{dt^\alpha}\left[\frac{\partial P(x)}{\partial x_k}\Big|_{\mathcal{J}_{s,\gamma}(t)}\right]_{t=0} \times (\mathcal{L}_{\mathtt{a}}^{-1}(\mathtt{y}))_{k,\ell} \, . \qquad (6.2.31)$$

By the expression above, it is immediate to check that if P satisfies the s-vanishing condition on $\mathcal{J}_{s,\gamma}$ then $\mathtt{P}_{\mathtt{a}}$ does the same on $\mathtt{J}_{s,\gamma,\mathtt{a}}$. The proof of the converse is immediate by applying the same arguments to $P(x) \equiv \mathtt{P}_{\mathtt{a}} \circ \mathcal{L}_{\mathtt{a}}(x)$. □

By the discussion above, one has the choice to write the equations determining the set $Z^1(r,s,m)$ in (6.2.11) either in the original coordinates, where they assume the form $q^1_{\ell\alpha} \circ \Phi(P, \mathcal{J}_{s,\gamma}) = 0$ for all $\ell \in \{1...,m\}$ and $\alpha \in \{0,...,s\}$, or in the new set of coordinates associated to the change of variables $\mathcal{L}_{\mathtt{a}}$ in $\mathbb{R}^m$, defined in (6.2.15). In particular, by performing the same computations that led to expression (6.2.7) in the new variables, and by taking into account the fact that the expansion of $\mathtt{J}_{s,\gamma,\mathtt{a}}(t)$ starts at order two in t, one can introduce the functions

$$\mathtt{Q}_{\ell\alpha} : W^1(r,m) \times \mathbb{R}^{(m-1)(s-1)} \longrightarrow \mathbb{R} \quad , \qquad \ell \in \{1,\dots,m\} \quad \alpha \in \{0,\dots,s\} \qquad (6.2.32)$$

in the following way:

For $\alpha = 0$, $\quad \mathtt{Q}_{\ell 0} \circ \Upsilon^1(P, \mathcal{J}_{s,\gamma}) = \mathtt{Q}_{\ell 0}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) := \mathtt{p}_{(0,\dots,0,1,0,\dots,0)}$

For $\alpha \in \{1,...,s\}$, $\quad \mathtt{Q}_{\ell\alpha} \circ \Upsilon^1(P, \mathcal{J}_{s,\gamma}) = \mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) := \dfrac{d^\alpha}{dt^\alpha}\left(\dfrac{\partial \mathtt{P}_{\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell}\Big|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)}\right)_{t=0}$

$$= \frac{d^\alpha}{dt^\alpha}\left[\sum_{\substack{\mu \in \mathbb{N}^m \\ 2 \le |\mu| \le r}} \mu_\ell \, \mathtt{p}_\mu t^{\tilde{\mu}_1(\ell)} \left(\sum_{k=2}^{s} a_{2k} t^k\right)^{\tilde{\mu}_2(\ell)} \cdots \left(\sum_{j=2}^{s} a_{mj} t^j\right)^{\tilde{\mu}_m(\ell)}\right]_{t=0} . \qquad (6.2.33)$$

Expressions (6.2.7) and (6.2.33), considered together with Lemma 6.2.2, imply that condition $q^1_{\ell\alpha} \circ \Phi^1(P, \mathcal{J}_{s,\gamma}) = 0$ holds if and only if

$$\mathtt{Q}_{\ell\alpha} \circ \Upsilon^1(P, \mathcal{J}_{s,\gamma}) = \mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0 \, , \quad \forall \ell \in \{1,\dots,m\}, \alpha \in \{0,\dots,s\} \, , \quad (6.2.34)$$

so that the ideal of the set $Z^1(r,s,m)$ in the new variables is given by the equations $\mathtt{Q}_{\ell\alpha} \circ \Upsilon^1(P, \mathcal{J}_{s,\gamma}) = \mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ for all $\ell \in \{1...,m\}$ and $\alpha \in \{0,...,s\}$. In the sequel, we will work in these new coordinates, since the involved expressions are nicer.

6.2.4 Computations and estimate on the codimension of $\sigma(r, s, m)$ (case $n \geq 3, 2 \leq m \leq n-1$)

As in the previous paragraphs, we set $n \geq 3$, $2 \leq m \leq n-1$.

Once again, we will only consider the case in which γ is parametrized by the first coordinate.

Before stating the main results of this paragraph, we still need to introduce a few notations. For any fixed $\alpha \in \{0, \dots, s\}$, for any $\beta \in \{0, \dots, \alpha\}$, and for any $i \in \{1, \dots, m\}$, we set

$$\nu(i, \beta) := \begin{cases} (\beta+1, 0, \dots, 0)\,, & \text{for } i = 1 \\ (\beta, 0, \dots, 0, 1, 0, \dots, 0)\,, & \text{for } i = 2, \dots, m \end{cases} \tag{6.2.35}$$

where the "1" fills the i-th slot for $i = 2, ..., m$. For $\alpha \in \{1, ..., s\}$, we also denote the multi-indices $\mu \in \mathbb{N}^m$ of length $2 \leq |\mu| \leq \alpha + 1$ not belonging to this family with

$$\mathcal{M}_m(\alpha) := \{\mu \in \mathbb{N}^m\,,\ \ 2 \leq |\mu| \leq \alpha + 1\} \setminus \bigcup_{\substack{i=1,\dots,m \\ \beta=1,\dots,\alpha}} \{\nu(i, \beta)\}\,. \tag{6.2.36}$$

Moreover, for any given $\alpha \in \{1, .., s\}$, $\mu \in \mathbb{N}^m$ and $\ell \in \{1, ..., m\}$ we introduce

$$\mathcal{G}_m(\widetilde{\mu}(\ell), \alpha) := \Bigg\{ (k_{j2}, ..., k_{j\alpha}) \in \mathbb{N}^{(m-1)\times(\alpha-1)}, j \in \{2, ..., m\} : \\ \sum_{i=2}^{\alpha} k_{ji} = \widetilde{\mu}_j(\ell)\,,\ \ \widetilde{\mu}_1(\ell) + \sum_{j=2}^{m} \sum_{i=2}^{\alpha} i\, k_{ji} = \alpha \Bigg\} \tag{6.2.37}$$

and we set

$$\mathcal{E}_m(\ell, \alpha) := \{\mu \in \mathbb{N}^m \mid \mathcal{G}_m(\widetilde{\mu}(\ell), \alpha) \neq \emptyset\}\,. \tag{6.2.38}$$

Finally, for any $\ell \in \{1, \dots, m\}$ and for any $\mu \in \mathbb{N}^m$, we remind that (see (6.2.5))

$$\widetilde{\mu}_j(\ell) := \mu_j - \delta_{j\ell} \quad j \in \{1, \dots, m\}\,.$$

With these notations, we can now state the following

Lemma 6.2.3. *For any choice of integers $m \geq 1$, $r \geq 2$, $1 \leq s \leq r-1$, the set $Z(r, s, m)$ in (6.2.12) is an algebraic set of $\mathcal{P}(r, m) \times \vartheta_m(s)$, whose ideal can be explicitly computed. In particular, with the notations in (6.2.33), (6.2.35), (6.2.36) and (6.2.37), the set $Z^1(r, s, m)$ is the image through the inverse of the transformation Υ^1 in (6.2.26)* *of the algebraic set determined by the following equations:*

$$
\begin{aligned}
&\mathtt{Q}_{10}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = \mathtt{p}_{\nu(1,0)} = 0\,,\\
&\mathtt{Q}_{11}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = 2\,\mathtt{p}_{\nu(1,1)} = 0\,,\\
&\frac{\mathtt{Q}_{1\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})}{\alpha!} = (\alpha+1)\mathtt{p}_{\nu(1,\alpha)} + \sum_{i=2}^{m}\sum_{\beta=1}^{\alpha-1}\beta\,\mathtt{p}_{\nu(i,\beta)}a_{i(\alpha-(\beta-1))}\\
&\qquad + \sum_{\substack{\mu\in\mathcal{E}_m(1,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_1\neq 0}} \mu_1\,\mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(1),\alpha)}\left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(1)}{k_{j2}\,\ldots\,k_{j\alpha}}\,{a_{j2}}^{k_{j2}}\ldots{a_{j\alpha}}^{k_{j\alpha}}\right) = 0\,,
\end{aligned}
\tag{6.2.39}
$$

for $\ell = 1$, $\alpha = 2,\ldots,s$, and

$$
\begin{aligned}
&\mathtt{Q}_{\ell 0}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = \mathtt{p}_{\nu(\ell,0)} = 0\,,\\
&\mathtt{Q}_{\ell 1}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = \mathtt{p}_{\nu(\ell,1)} = 0\,,\\
&\frac{\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})}{\alpha!} = \mathtt{p}_{\nu(\ell,\alpha)}\\
&\qquad + \sum_{\substack{\mu\in\mathcal{E}_m(\ell,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_\ell\neq 0}} \mu_\ell\,\mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)}\left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(\ell)}{k_{j2}\,\ldots\,k_{j\alpha}}\,{a_{j2}}^{k_{j2}}\ldots{a_{j\alpha}}^{k_{j\alpha}}\right) = 0\,,
\end{aligned}
$$

for $\ell = 2,\ldots,m\,,\ \alpha = 2,\ldots,s\,.$

(6.2.40)

Remark 6.2.6. It is plain to check that the coefficients of the vector $\mathtt{a}\in\vartheta^1_m(1)$ containing the linear terms of the truncation $\mathcal{J}_{s,\gamma}$ do not appear explicitly in expressions (6.2.39)-(6.2.40). However, they are "hidden" in the terms $\mathtt{p}_\mu = \mathtt{p}_\mu(p_\mu,\mathtt{a})$ (see (6.2.16)).

As an almost immediate consequence of Lemma 6.2.3, we have that s-vanishing polynomials are rare in $\mathcal{P}(r,m)$, namely

Corollary 6.2.1. *$Z(r,s,m)$ has codimension $m(s+1)$ in $\mathcal{P}(r,m)\times\vartheta_m(s)$ and $\sigma(r,s,m)$ is a semi-algebraic set of codimension $s+m$ in $\mathcal{P}(r,m)$.*

Proof. (Lemma 6.2.3) For fixed $s\in\{1,\ldots,r-1\}$, we consider a polynomial $P\in\mathcal{P}(r,m)$ verifying the s-vanishing condition on some truncation $\mathcal{J}_{s,\gamma}(t)\in\vartheta^1_m(s)$.

Step 1. By Lemma 6.2.2, in the adapted coordinates (6.2.15) one must have

$$
\frac{d^\alpha}{dt^\alpha}\left(\left.\frac{\partial\mathtt{P}_{\mathtt{a}}(\mathtt{y})}{\partial\mathtt{y}_\ell}\right|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)}\right)_{t=0} = 0 \qquad \forall\alpha\in\{0,\ldots,s\}\,,\ \ \forall\ell\in\{1,\ldots,m\}\,. \tag{6.2.41}
$$

For $\alpha = 0$, it is plain to check that the terms of order zero in t in (6.2.41) are the linear terms of $\mathtt{P}_{\mathtt{a}}$, for which $|\mu| = 1$. Expressions (6.2.41) and (6.2.33) yield the thesis for this value of α.

Then, as we did in (6.2.5), we drop the linear terms in $\mathtt{P_a}$ and we write the quantity $\partial \mathtt{P_a}(\mathtt{y})/\partial \mathtt{y}_\ell$ explicitly

$$\frac{\partial \mathtt{P_a}(\mathtt{y})}{\partial \mathtt{y}_\ell} := \sum_{\substack{\mu\in\mathbb{N}^m \\ 2\le|\mu|\le r}} \mu_\ell \, \mathtt{p}_\mu \mathtt{y}^{\widetilde{\mu}(\ell)} \;,\;\; \widetilde{\mu}_j(\ell) := \mu_j - \delta_{j\ell} \;,\;\; j = 1, ..., m \;,\;\; |\mu| = |\widetilde{\mu}(\ell)| + 1 \;. \tag{6.2.42}$$

Taking (6.2.17) into account (with $\mathtt{b} := (b_{21}, \dots, b_{m1}) = \mathtt{a} := (a_{21}, \dots, a_{m1})$), we inject in (6.2.42) the components of the s-truncation $\mathtt{J}_{s,\gamma,\mathtt{a}}(t)$, namely

$$\mathtt{y}_1(t) = t \quad , \qquad \mathtt{y}_j(t) = \sum_{i=2}^{s} a_{ji} t^i \quad , \qquad j \in \{2, ..., m\} \;, \tag{6.2.43}$$

and we obtain

$$\left.\frac{\partial \mathtt{P_a}(\mathtt{y})}{\partial \mathtt{y}_\ell}\right|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} = \sum_{\substack{\mu\in\mathbb{N}^m \\ 2\le|\mu|\le r}} \mu_\ell \, \mathtt{p}_\mu t^{\widetilde{\mu}_1(\ell)} \left(\sum_{i=2}^{s} a_{2i} t^i\right)^{\widetilde{\mu}_2(\ell)} \cdots \left(\sum_{u=2}^{s} a_{mu} t^u\right)^{\widetilde{\mu}_m(\ell)} \cdot \tag{6.2.44}$$

Step 2. For $\alpha = 1$, we must look for the coefficient of the linear term (in t) in expression (6.2.44). Hence, for fixed $\ell = 1, ..., m$, since the sums in (6.2.44) start at order two in t, only the multi-index for which $\widetilde{\mu}_j(\ell) = 0$ for all $j \in \{2, ..., m\}$ and $\widetilde{\mu}_1(\ell) = 1$ must be retained in the sum in expression (6.2.44). The first condition implies $\mu_j(\ell) = \delta_{j\ell}$ for all $j \in \{2, ..., m\}$, whereas the second yields $\mu_1(\ell) = 1 + \delta_{1\ell}$. Therefore, by definition (6.2.35), for fixed $\ell \in \{1, ..., m\}$ only the multi-index $\nu(\ell, 1)$ appears in expression (6.2.44) for $\alpha = 1$. Again by (6.2.35), one has $\nu_1(1, 1) = 2$ and $\nu_\ell(\ell, 1) = 1$ for $\ell \in \{2, ..., m\}$ so that the thesis in the case $\alpha = 1$ follows.

Step 3. For any given $\alpha \in \{2, ..., s\}$, we are interested in the coefficients of the terms of order t^α in (6.2.44). Hence, we can truncate the internal sums in (6.2.44) at order α. For the same reason, for any $\ell \in \{1, ..., m\}$, we can neglect from the leftmost sum in (6.2.44) the monomials μ satisfying $|\widetilde{\mu}(\ell)| > \alpha$ (hence $|\mu| > \alpha + 1$), as their contribution is of order at least $t^{\alpha+1}$. Thus, for fixed $\alpha \in \{2, ..., s\}$ and $\ell \in \{1, ..., m\}$, we have

$$\begin{aligned} &\mathtt{Q}_{\ell\alpha}(\mathtt{P_a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) \\ &= \frac{d^\alpha}{dt^\alpha} \left(\sum_{\substack{\mu\in\mathbb{N}^m \\ 2\le|\mu|\le\alpha+1}} \mu_\ell \mathtt{p}_\mu t^{\widetilde{\mu}_1(\ell)} \left(\sum_{i=2}^{\alpha} a_{2i} t^i\right)^{\widetilde{\mu}_2(\ell)} \cdots \left(\sum_{u=2}^{\alpha} a_{mu} t^u\right)^{\widetilde{\mu}_m(\ell)} \right)_{t=0} \end{aligned} \tag{6.2.45}$$

due to formula (6.2.33). Now, for any $j = 2, \dots, m$ and $\ell \in \{1, \dots, m\}$, the multinomial expansion yields:

$$\left(\sum_{i=2}^{\alpha} a_{ji} t^i\right)^{\widetilde{\mu}_j(\ell)} = \sum_{\substack{k_{j2},\ldots,k_{j\alpha}\in\mathbb{N} \\ k_{j2}+\ldots+k_{j\alpha}=\widetilde{\mu}_j(\ell)}} \binom{\widetilde{\mu}_j(\ell)}{k_{j2}\ \ldots\ k_{j\alpha}} {a_{j2}}^{k_{j2}} \ldots {a_{j\alpha}}^{k_{j\alpha}}\ t^{2k_{j2}+\ldots+\alpha k_{j\alpha}}, \tag{6.2.46}$$

where we have used the notation $\binom{\widetilde{\mu}_j(\ell)}{k_{j2}\ \ldots\ k_{j\alpha}} := \frac{\widetilde{\mu}_j(\ell)!}{k_{j2}!\ \ldots\ k_{j\alpha}!}$.

Replacing each truncated Taylor development in (6.2.45) by its multinomial expansion (6.2.46), expression (6.2.45) reads

$$\frac{d^\alpha}{dt^\alpha}\left[\sum_{\substack{\mu\in\mathbb{N}^m \\ 2\le|\mu|\le\alpha+1}} \mu_\ell\ \mathtt{p}_\mu\ t^{\widetilde{\mu}_1(\ell)} \prod_{j=2}^{m}\left(\sum_{\substack{k_{j2},\ldots,k_{j\alpha}\in\mathbb{N} \\ k_{j2}+\ldots+k_{j\alpha}=\widetilde{\mu}_j(\ell)}} \binom{\widetilde{\mu}_j(\ell)}{k_{j2}\ \ldots\ k_{j\alpha}} {a_{j2}}^{k_{j2}}\ldots{a_{j\alpha}}^{k_{j\alpha}}\ t^{2k_{j2}+\ldots+\alpha k_{j\alpha}}\right)\right]_{t=0} =$$

$$\frac{d^\alpha}{dt^\alpha}\left[\sum_{\substack{\mu\in\mathbb{N}^m \\ 2\le|\mu|\le\alpha+1}} \mu_\ell\ \mathtt{p}_\mu\ t^{\widetilde{\mu}_1(\ell)} \sum_{\substack{k\in\mathbb{N}^{(m-1)\times(\alpha-1)} \\ k=(k_{22},\ldots,k_{2\alpha},\ldots,k_{m2},\ldots,k_{m\alpha}) \\ \forall i\in\{2,\ldots,m\} \\ k_{i2}+\ldots+k_{i\alpha}=\widetilde{\mu}_i(\ell)}} \prod_{j=2}^{m}\binom{\widetilde{\mu}_j(\ell)}{k_{j2}\ \ldots k_{j\alpha}} {a_{j2}}^{k_{j2}}\ldots{a_{j\alpha}}^{k_{j\alpha}}\ t^{2k_{j2}+\ldots+\alpha k_{j\alpha}}\right]_{t=0}. \tag{6.2.47}$$

Moreover, taking (6.2.37) into account, the class of multi-indices $\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)$ selects those terms whose contribution inside the brackets of (6.2.47) is of order t^α. Hence, by the above discussion, by (6.2.33) and by (6.2.38), for any fixed $\alpha\in\{2,\ldots,s\}$, and $\ell\in\{1,\ldots,m\}$, we can write

$$\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = \alpha! \sum_{\substack{\mu\in\mathcal{E}_m(\ell,\alpha) \\ \mu_\ell\neq 0}} \mu_\ell\ \mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)} \left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(\ell)}{k_{j2}\ \ldots\ k_{j\alpha}} {a_{j2}}^{k_{j2}}\ldots{a_{j\alpha}}^{k_{j\alpha}}\right). \tag{6.2.48}$$

Now, we split the leftmost sum in (6.2.48) into the partial sums with respect to the families of indices defined in (6.2.35) and (6.2.36), namely for any fixed $\alpha\in\{2,\ldots,s\}$

and $\ell \in \{1, ..., m\}$, we write

$$\frac{\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})}{\alpha!} = \sum_{\substack{i=1 \\ \nu_\ell(i,\beta)\in\mathcal{E}_m(\ell,\alpha) \\ \nu_\ell(i,\beta)\neq 0}}^{m} \sum_{\beta=1}^{\alpha} \nu_\ell(i,\beta)\, \mathtt{p}_{\nu(i,\beta)} \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)} \left(\prod_{j=2}^{m} \binom{\widetilde{\nu}_j(i,\beta)(\ell)}{k_{j2} \ \dots \ k_{j\alpha}} {a_{j2}}^{k_{j2}} \dots {a_{j\alpha}}^{k_{j\alpha}} \right)$$
$$+ \sum_{\substack{\mu\in\mathcal{E}_m(\ell,\alpha) \\ \mu\in\mathcal{M}_m(\alpha) \\ \mu_\ell\neq 0}} \mu_\ell\, \mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)} \left(\prod_{j=2}^{m} \binom{\widetilde{\mu}_j(\ell)}{k_{j2} \ \dots \ k_{j\alpha}} {a_{j2}}^{k_{j2}} \dots {a_{j\alpha}}^{k_{j\alpha}} \right). \tag{6.2.49}$$

Step 4. We first study the case in which $\ell \neq 1$. For fixed $\ell \in \{2, ..., m\}$, for any $i \in \{1, ..., m\}$, $i \neq \ell$, and for any $\beta \in \{1, ..., \alpha\}$, the monomials corresponding to the indices $\nu(i, \beta)$ do not contribute to the leftmost sum at the r.h.s. of (6.2.49). Infact, by (6.2.35), the ℓ-th element $\nu_\ell(i, \beta)$ of multi-index $\nu(i, \beta)$ is equal to zero for $\ell \in \{2, ..., m\}$ and $i \in \{1, ..., m\}$, $i \neq \ell$.

Moreover, still for fixed $\ell \in \{2, ..., m\}$, the indices $\nu(\ell, \beta)$, with $\beta \in \{1, ..., \alpha\}$, satisfy $\widetilde{\nu}_1(\ell, \beta)(\ell) = \beta$ and $\widetilde{\nu}_j(\ell, \beta)(\ell) = 0$ for all $j \in \{2, ..., m\}$, so that by (6.2.37) we have

$$\mathcal{G}_m(\widetilde{\nu}(\ell, \beta)(\ell), \alpha) = \begin{cases} \varnothing, & \text{if } \beta = 1, ..., \alpha - 1 \\ \{0\}, & \text{if } \beta = \alpha \end{cases}.$$

Consequently, the only monomial that contributes to the leftmost sum at the r.h.s. of (6.2.49) is the one associated to the multi-index $\nu(\ell, \alpha)$, and one has $k_{j2} = 0, \dots, k_{j\alpha} = 0$ when $\mu = \nu(\ell, \alpha)$. Moreover, by hypothesis we have $\nu_\ell(\ell, \alpha) = 1$ for any $\ell \in \{2, ..., m\}$. Due to these arguments, for any fixed $\ell \in \{2, ..., m\}$, we can rewrite (6.2.49) in the form

$$\frac{\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})}{\alpha!} = \mathtt{p}_{\nu(\ell,\alpha)} + \sum_{\substack{\mu\in\mathcal{E}_m(\ell,\alpha) \\ \mu\in\mathcal{M}_m(\alpha) \\ \mu_\ell\neq 0}} \mu_\ell\, \mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(\ell),\alpha)} \left(\prod_{j=2}^{m} \binom{\widetilde{\mu}_j(\ell)}{k_{j2} \ \dots \ k_{j\alpha}} {a_{j2}}^{k_{j2}} \dots {a_{j\alpha}}^{k_{j\alpha}} \right). \tag{6.2.50}$$

This proves the Lemma for $\ell = 2, ..., m$, $\alpha = 2, ..., s$.

Step 5. We now consider the case $\ell = 1$. For all $j \in \{2, ..., m\}$, the sub-family of indices $\nu(1, \beta)$, with $\beta \in \{1, ..., \alpha\}$, satisfies $\widetilde{\nu}_1(1, \beta)(1) = \beta$ and $\widetilde{\nu}_j(1, \beta)(1) = 0$. Hence, thanks to (6.2.37), we find

$$\mathcal{G}_m(\widetilde{\nu}(1, \beta)(1), \alpha) = \begin{cases} \varnothing, & \text{if } \beta = 1, ..., \alpha - 1 \\ \{0\}, & \text{if } \beta = \alpha \end{cases}. \tag{6.2.51}$$

Moreover, we have $\nu_1(1,\alpha) = \alpha + 1$ by construction.

On the other hand, for $\ell = 1$, $\beta \in \{1, ..., \alpha\}$ and $i, j \in \{2, ..., m\}$, the multi-indices $\nu(i,\beta)$ satisfy $\widetilde{\nu}_1(i,\beta)(1) = \beta - 1$ and $\widetilde{\nu}_j(i,\beta)(1) = \delta_{ji}$. Hence, by (6.2.37) one can write

$$\widetilde{\nu}_j(i,\beta)(1) = \delta_{ji} = \sum_{u=2}^{\alpha} k_{ju} \iff k_{ju} = \delta_{ji}\delta_{uv} \text{ for some } v \in \{2, ..., \alpha\} \tag{6.2.52}$$

and

$$\widetilde{\nu}_1(i,\beta)(1) + \sum_{j=2}^{m}\sum_{u=2}^{\alpha} u\, k_{ju} = \beta - 1 + \sum_{j=2}^{m}\sum_{u=2}^{\alpha} u\, k_{ju} = \alpha\ . \tag{6.2.53}$$

By (6.2.52), we see that condition (6.2.53) can be satisfied by some vector of multi-integers $(k_{22}, \ldots, k_{2\alpha}, \ldots, k_{m2}, \ldots, k_{m\alpha})$ if $\beta \in \{1, ..., \alpha - 1\}$, but cannot be fulfilled for $\beta = \alpha$. Injecting (6.2.52) into (6.2.53) one has

$$\sum_{j=2}^{m}\sum_{u=2}^{\alpha} u\, \delta_{ji}\delta_{uv} = \alpha - (\beta - 1) \quad \text{for all} \quad \beta \in \{1, ..., \alpha - 1\}\ ,\ \ i \in \{2, ..., m\} \tag{6.2.54}$$

which implies

$$k_{ju} = \delta_{ji}\,\delta_{uv}\,\delta_{v,\alpha-(\beta-1)} \quad \text{for all} \quad \beta \in \{1, ..., \alpha - 1\}\ ,\ \ i \in \{2, ..., m\}\ . \tag{6.2.55}$$

Hence, for all $\beta \in \{1, ..., \alpha - 1\}$, and $i \in \{2, ..., m\}$, we can finally write

$$\mathcal{G}_m(\widetilde{\nu}(i,\beta)(1),\alpha) = \{(k_{j1}, ..., k_{j\alpha}),\ j \in \{2, ..., m\},\ k_{ju} = \delta_{ji}\delta_{u,\alpha-(\beta-1)}\} \tag{6.2.56}$$

and

$$\mathcal{G}_m(\widetilde{\nu}(i,\alpha)(1),\alpha) = \varnothing\ . \tag{6.2.57}$$

Moreover, for $i = 2, ..., m$, by (6.2.35) we have $\nu_1(i,\beta) = \beta$.

By taking (6.2.51), (6.2.56), (6.2.57) into account, expression (6.2.49) with $\ell = 1$ yields

$$\begin{aligned}\frac{\mathtt{Q}_{1\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})}{\alpha!} &= (\alpha+1)\mathtt{p}_{\nu(1,\alpha)} + \sum_{i=2}^{m}\sum_{\beta=1}^{\alpha-1} \beta\ \mathtt{p}_{\nu(i,\beta)} a_{i(\alpha-(\beta-1))} \\ &+ \sum_{\substack{\mu\in\mathcal{E}_m(1,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_1\neq 0}} \mu_1\ \mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(1),\alpha)} \left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(1)}{k_{j2}\ \ldots\ k_{j\alpha}} {a_{j2}}^{k_{j2}} \ldots {a_{j\alpha}}^{k_{j\alpha}}\right)\ .\end{aligned} \tag{6.2.58}$$

This concludes the proof for the case in which $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$.

The proof of the case in which $\mathcal{J}_{s,\gamma} \in \vartheta^i_m(s)$, with $i = 2, \ldots, m$, is the same: one just has to take into account that the rôle of the special index is played by i instead of 1. Hence, the ideals of the sets $Z^i(r,s,m)$ can be explicitly computed and, by expression (6.2.12), the proof is concluded.

□

We are now able to prove that s-vanishing polynomials are rare in $\mathcal{P}(r, m)$.

Proof. (Corollary 6.2.1) We want to show that, for a given pair $(P, \mathcal{J}_{s,\gamma}) \in \mathcal{P}(r, m) \times \vartheta^1_m(s)$, the $ms + m$ equations in (6.2.39) and (6.2.40) are all linearly independent.

For all $\ell = 1, ..., m$ and $\alpha = 0, ..., s$, we collect in table (6.2.60) the derivatives of the functions $\mathtt{Q}_{\ell\alpha}$ defined in (6.2.32)-(6.2.33) - and whose action is made explicit in (6.2.39) - (6.2.40) - with respect to the coefficients of $\mathtt{P}_{\mathtt{a}}$, $\mathtt{a}$, and to the Taylor coefficients of $\mathtt{J}_{s,\gamma,\mathtt{a}}$. We have indicated

1. with the symbol $\mathbb{D}$, the $s \times s$ diagonal matrix whose entries are the numbers $\alpha + 1$, for $\alpha = 1, ..., s$;

2. with the symbol $\mathbb{I}_s$, the $s \times s$ identity matrix;

3. with the symbol $\mathbb{B}_i$, $i \in \{2, ..., m\}$, an $s \times s$ matrix whose entry at position α, β, with $\alpha \in \{1, ..., s\}$ and $\beta \in \{1, ..., s\}$, reads

$$(\mathbb{B}_i)_{\alpha,\beta} := \begin{cases} 0\,, & \text{if } \alpha = 1\,, \\ \beta a_{i(\alpha-(\beta-1))}\,, & \text{if } 2 \le \alpha \le s\,,\ \ 1 \le \beta \le \alpha - 1\,, \\ 0\,, & \text{if } 2 \le \alpha \le s\,,\ \ \alpha \le \beta \le s\,. \end{cases} \tag{6.2.59}$$

ℓ	α	$\partial_{a_{ji}}$	$\partial_{\mathtt{p}_\mu}$ $\mu \in \mathcal{M}(s)$	$\partial_{\mathtt{p}_{\nu(i,0)}}$ $i = 1, \dots, m$	$\partial_{\mathtt{p}_{\nu(1,\beta)}}$ $\beta = 1, \dots, s$	$\partial_{\mathtt{p}_{\nu(2,\beta)}}$ $\beta = 1, \dots, s$	$\dots$ $\dots$	$\partial_{\mathtt{p}_{\nu(m,\beta)}}$ $\beta = 1, \dots, s$
$1, \dots, m$	0	0	0	$\mathbb{I}_m$	0	0	0	0
1	$1, \dots, s$	$\dots$	$\dots$	0	$\mathbb{D}$	$\mathbb{B}_2$	$\dots$	$\mathbb{B}_m$
2	$1, \dots, s$	$\dots$	$\dots$	0	0	$\mathbb{I}_s$	0	0
$\dots$	$\dots$	$\dots$	$\dots$	0	0	0	$\dots$	0
m	$1, \dots, s$	$\dots$	$\dots$	0	0	0	0	$\mathbb{I}_s$

(6.2.60)

Table 6.2.60: Jacobian of $\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ with $\ell \in \{1, ..., m\}$ and $\alpha = \{0, .., s\}$. The first and the second column contain, respectively, all the possible values for the parameters ℓ and α. The third column corresponds to the derivatives of $\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})$ with respect to the variables of the vector $\mathtt{a} \in \vartheta^1_m(1)$, and to the Taylor coefficients of the s-jet $\mathtt{J}_{s,\gamma,\mathtt{a}}$. The remaining columns contain the derivatives with respect to the coefficients $\mathtt{p}_\mu$ of $\mathtt{P}_{\mathtt{a}}$ associated with the families of multi-indices (6.2.35) and (6.2.36), in suitable order.

It is plain to check that matrix (6.2.60) contains a submatrix of maximal rank $ms+m$ - corresponding to the derivatives w.r.t. those coefficients associated to the family of multi-indices (6.2.35) - independently of $\mathtt{a}$ and of the s-truncation $\mathtt{J}_{s,\gamma,\mathtt{a}}$ on which the s-vanishing condition is realized. Hence, since the transformation Υ in (6.2.25) is invertible, by Lemma 6.2.3 the set $Z^1(r,s,m)$ is determined by $ms+m$ linearly independent algebraic equations and has codimension $ms+m$ in $\mathcal{P}(r,m)\times\vartheta^1_m(s)$.

As it was the case in the proof of Lemma 6.2.3, the same strategy of proof applies for $Z^j(r,s,m)$, $j=2,\dots,m$, one just has to switch the rôle of the indices 1 and j.

Since $\sigma^1(r,s,m) := \Pi_{\mathcal{P}(r,m)}Z^1(r,s,m)$ (see (6.2.10)) and $Z^1(r,s,m)$ is algebraic, by the Theorem of Tarski and Seidenberg (see Th. A.1.1) $\sigma^1(r,s,m)$ is a semi-algebraic set of $\mathcal{P}(r,m)$. Moreover, as Jacobian (6.2.60) has rank $ms+m$ w.r.t. the $ms+m$ polynomial coefficients associated to the multi-indices of the family (6.2.35), for $\alpha\in\{2,...,s\}$ and $i\in\{1,...,m\}$, by Lemma 6.2.3 and by the implicit function theorem, the conditions $\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})=0$ imply

$$\begin{aligned}&\mathtt{p}_{\nu(i,0)}=\mathtt{p}_{\nu(i,1)}=0\\&\mathtt{p}_{\nu(i,\alpha)}=g_{i\alpha}(\mathtt{p}_\mu,\mathtt{a},a_{22},...,a_{2s},...,a_{m2},...,a_{ms})\quad,\qquad\mu\in\mathcal{M}(\alpha)\end{aligned}\tag{6.2.61}$$

for some implicit functions $g_{i\alpha}$. That is, one can express the polynomial coefficients $\mathtt{p}_{\nu(i,0)},\mathtt{p}_{\nu(i,1)},\mathtt{p}_{\nu(i,\alpha)}$ as implicit functions of the remaining coefficients - associated to the multi-indices in the family $\mathcal{M}(\alpha)$ defined in (6.2.35) - and of the $(m-1)s$ parameters of $\mathtt{a}$ and $\mathtt{J}_{s,\gamma,\mathtt{a}}$. Moreover, since the functions $\mathtt{Q}_{\ell\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})$ are polynomial for all $\ell\in\{1,...,m\}$ and $\alpha\in\{0,..,s\}$, the implicit functions $g_{i\alpha}$ are all analytic. Therefore, one has an analytic parametrization of $\sigma^1(r,s,m)$ given by the $m(s+1)$ independent equations (6.2.61), for $i\in\{1,...,m\}$, $\alpha\in\{0,...,s\}$. This, in turn, yields that

$$\begin{aligned}\dim\sigma^1(r,s,m)&=\dim W^1(r,m)-m(s+1)=\dim\mathcal{P}(r,m)+\dim\vartheta^1_m(s)-m(s+1)\\&=\dim\mathcal{P}(r,m)+(m-1)s-m(s+1)\,,\end{aligned}\tag{6.2.62}$$

which implies that the codimension of $\sigma^1(r,s,m)$ in $\mathcal{P}(r,m)$ is

$$\operatorname{codim}\sigma^1(r,s,m)=m(s+1)-(m-1)s=m+s\,.\tag{6.2.63}$$

Once again, it is plain to check that the same result holds true also in the case in which the parametrizing coordinate of the curve gamma is the j-th, with $j=2,\dots,m$. Hence, one finds codim $\sigma^j(r,s,m)=m+s$ for $j=2,\dots,m$, which, together with expression (6.2.9), proves the statement.

□

6.3 Geometric properties

For fixed integers $r\geq 2$, $m\geq 2$, $1\leq s\leq r-1$, and for any $i\in\{1,\dots,m\}$, we indicate respectively by $\Sigma(r,s,m) := \overline{\sigma}(r,s,m)$ and $\Sigma^i(r,s,m) := \overline{\sigma}^i(r,s,m)$ the closures in

$\mathcal{P}(r,m)$ of the sets $\sigma(r,s,m)$ and $\sigma^i(r,s,m)$ introduced in the previous section. Taking (6.2.9) into account, one has

$$\Sigma(r,s,m) = \bigcup_{i=1}^{m} \Sigma^i(r,s,m) \ . \tag{6.3.1}$$

For $m \geq 2$, Corollary 6.2.1 and Proposition A.1.3 ensure that for any $i \in \{1,\dots,m\}$

$$\text{codim } \Sigma^i(r,s,m) = \text{ codim } \sigma^i(r,s,m) = s + m > 0$$

in $\mathcal{P}(r,m)$, so that $\mathcal{P}(r,m)\backslash\Sigma^i(r,s,m)$ is an open set of full Lebesgue measure. Therefore, by (6.3.1), the same holds true also for $\mathcal{P}(r,m)\backslash\Sigma(r,s,m)$.

As we did previously, when $m \geq 2$ we only consider the case in which the index of the parametrizing coordinate is $i = 1$, as the other cases are immediate generalizations.

In case $m = 1$, Lemma 6.2.1 ensures that

$$\sigma(r,s,1) = \overline{\sigma}(r,s,1) =: \Sigma(r,s,1) \ . \tag{6.3.2}$$

and

$$\text{codim } \sigma(r,s,1) = \text{codim } \Sigma(r,s,1) = s+1 \ . \tag{6.3.3}$$

Still for $m = 1$, in order to make use of uniform notations w.r.t. the case $m = 2$ and to simplify the exposition in the sequel, we write

$$\sigma^1(r,s,m) \equiv \Sigma^1(r,s,1) := \Sigma(r,s,1) \tag{6.3.4}$$

and we extend the notations of subsection 6.2.3 by setting

$$\mathcal{L}_{\mathtt{a}} := \text{id} \quad , \qquad \mathtt{P}_{\mathtt{a}}(\mathtt{y}) := P(\mathtt{y}) \quad , \qquad \text{for } m = 1 \ . \tag{6.3.5}$$

The rest of this section will be devoted to proving the following

Lemma 6.3.1. *Let m be a positive integer. For any open set $\mathtt{D} \subset \mathcal{P}(r,m)\backslash\Sigma^1(r,s,m)$ verifying $\overline{\mathtt{D}} \cap \Sigma^1(r,s,m) = \emptyset$, there exist positive constants $C_1(\mathtt{D})$, $C_2(s,m)$ such that for any polynomial $P(x) \in \mathtt{D}$ and for any arc $\gamma \in \Theta^1_m$ one has the following lower estimates*

$$\begin{aligned}
&\max_{\substack{\ell=1,\dots,m\\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial \mathtt{P}_{\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell}\bigg|_{\mathcal{L}_{\mathtt{a}}\circ\gamma(t)} \right)_{t=0} \right| > C_1(\mathtt{D}) \\
&\textit{in case } s = 1 \textit{ or } \ m = 1, \\
&\max_{\substack{\ell=1,\dots,m\\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial \mathtt{P}_{\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell}\bigg|_{\mathcal{L}_{\mathtt{a}}\circ\gamma(t)} \right)_{t=0} \right| > \frac{C_1(\mathtt{D})}{1 + C_2(s,m) \times \max_{\substack{\ell=2,\dots,m\\ \alpha=2,\dots,s}} |a_{\ell\alpha}|} \\
&\textit{in case } 2 \leq s \leq r-1 \textit{ and } \ m \geq 2 \ ,
\end{aligned} \tag{6.3.6}$$

where - for $m \geq 2$ - $\mathtt{P}_{\mathtt{a}}$ is the polynomial P written in the adapted system of coordinates for γ introduced in paragraph 6.2.3 and $\mathcal{L}_{\mathtt{a}}$ is the associated transformation defined in (6.2.29), whereas for $m = 1$ the symbols $\mathtt{P}_{\mathtt{a}}$ and $\mathcal{L}_{\mathtt{a}}$ are to be intended as in (6.3.5).

Remark 6.3.1. As we shall see in the next section, for any $\lambda > 0$, when γ is the minimal arc of Theorem 5.0.1, one can give a positive upper bound - that only depends on r, s, m, λ - to the quantity $\max_{\substack{\ell=1,\dots,m \\ \alpha=2,\dots,s}} |a_{\ell\alpha}|$ at the denominator of (6.3.6). This is due to the fact that all minimal arcs satisfy a uniform Bernstein-like inequality on their Taylor coefficients (see formula (5.0.1) in Theorem 5.0.1).

Before proving Lemma 6.3.1, we need an intermediate result and a few additional notations.

In case $m \geq 2$, for any given arc $\gamma \in \Theta^1_m$ with associated change of coordinates $\mathcal{L}_{\mathtt{a}}$ (see paragraph 6.2.3, in particular formulas (6.2.15)-(6.2.29)), we define the direct sum

$$\mathcal{P}(r,m) = \mathcal{P}^\sharp_{\mathtt{a}}(r,m) \oplus \mathcal{P}^\flat_{\mathtt{a}}(r,m)$$

in the following way: for any polynomial $R(x) \in \mathcal{P}(r,m)$, we consider its expression $\mathtt{R}_{\mathtt{a}}(\mathtt{y}) := R \circ \mathcal{L}^{-1}_{\mathtt{a}}(\mathtt{y}) \in \mathcal{P}(r,m)$ in the adapted coordinates (6.2.15) for γ; $\mathtt{R}_{\mathtt{a}}(\mathtt{y})$ can be decomposed uniquely into the partial sums with respect to the families of multi-indices defined in (6.2.35) and (6.2.36), namely:

$$\mathtt{R}_{\mathtt{a}}(\mathtt{y}) = \sum_{\substack{\mu \in \mathbb{N}^m \\ 1 \leq |\mu| \leq r}} \mathtt{r}_\mu \mathtt{y}^\mu = \sum_{i=1}^{m} \sum_{\beta=0}^{s} \mathtt{r}_{\nu(i,\beta)} \mathtt{y}^{\nu(i,\beta)} + \sum_{\substack{\mu \in \mathbb{N}^m \\ \mu \in \mathcal{M}(s)}} \mathtt{r}_\mu \mathtt{y}^\mu =: \mathtt{R}^\sharp_{\mathtt{a}}(\mathtt{y}) + \mathtt{R}^\flat_{\mathtt{a}}(\mathtt{y}) \,, \quad (6.3.7)$$

and we set $R^\sharp(x) := \mathtt{R}^\sharp_{\mathtt{a}} \circ \mathcal{L}_{\mathtt{a}}(x) \in \mathcal{P}^\sharp_{\mathtt{a}}(r,m)$ and $R^\flat(x) := \mathtt{R}^\flat_{\mathtt{a}} \circ \mathcal{L}_{\mathtt{a}}(x) \in \mathcal{P}^\flat_{\mathtt{a}}(r,m)$. Clearly, the decomposition $R(x) = R^\sharp(x) + R^\flat(x)$ is unique, as the function associating $R \longmapsto \mathtt{R}_{\mathtt{a}} := R \circ \mathcal{L}^{-1}_{\mathtt{a}}$, with $R \in \mathcal{P}(r,m)$, is invertible.

Finally, we set

$$\begin{aligned} &\mathtt{Q} \circ \Upsilon^1 \,:\, \mathcal{P}(r,m) \times \vartheta^1_m(s) \longrightarrow \mathbb{R}^{m(s+1)} \\ &(R, \mathcal{J}_{s,\gamma}) \longmapsto \mathtt{Q}_{\ell\alpha} \circ \Upsilon^1(R, \mathcal{J}_{s,\gamma}) \equiv \mathtt{Q}_{\ell\alpha}(\mathtt{R}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) \,, \end{aligned} \quad (6.3.8)$$

where $\ell = 1, \dots, m$, $\alpha = 0, \dots, s$, $\mathcal{J}_{s,\gamma}$ is the s-truncation of the curve γ, and the explicit form of $\mathtt{Q}_{\ell\alpha} \circ \Upsilon^1$ is given in Lemma 6.2.3. We also indicate by $\mathcal{N}(\cdot)$ the zero sets of the transformations which will appear henceforth.

With this setting, one has the following result:

Lemma 6.3.2. *In case $m \geq 2$, for any given $\gamma \in \Theta^1_m$ with associated s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$, and for any given polynomial $R(x) \in \mathcal{P}(r,m) \backslash \Sigma^1(r,s,m)$, there exists a unique polynomial $S(x) \in \sigma^1(r,s,m)$ such that*

$$(S, \mathcal{J}_{s,\gamma}) \in \mathcal{N}(\mathtt{Q} \circ \Upsilon^1) \quad , \qquad S^\flat = R^\flat \,.$$

In particular, S satisfies the s-vanishing condition on the truncation $\mathcal{J}_{s,\gamma}$.

Proof. Given $\gamma \in \Theta^1_m$ with its associated s-truncation $\mathcal{J}_{s,\gamma} \in \vartheta^1_m(s)$ and a polynomial $R(x) \in \mathcal{P}(r,m) \backslash \Sigma^1(r,s,m)$, we denote by

$$\mathtt{A}^\sharp_{R^\flat, \mathcal{J}_{s,\gamma}} \,:\, \Upsilon^1(\mathcal{P}^\sharp_{\mathtt{a}} \oplus \{R^\flat\} \times \{\mathcal{J}_{s,\gamma}\}) \to \mathbb{R}^{m(s+1)} \quad (6.3.9)$$

the restriction of $\mathtt{Q}$ to the set $\Upsilon(\mathcal{P}^\sharp_{\mathtt{a}} \oplus \{R^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$.

As it was shown in the proof of Corollary 6.2.1 (see Table (6.2.60)), for all $\alpha \in \{0, ..., s\}$ and $\ell \in \{1, ..., m\}$, the Jacobian matrix of $\mathtt{A}^\sharp_{R^\flat,\mathcal{J}_{s,\gamma}}$ reads

$$\mathcal{A} := \begin{pmatrix} \mathbb{I}_m & 0 & 0 & 0 & ... & 0 \\ 0 & \mathbb{D} & \mathbb{B}_2 & \mathbb{B}_3 & ... & \mathbb{B}_m \\ 0 & 0 & \mathbb{I}_s & 0 & ... & 0 \\ 0 & 0 & 0 & \mathbb{I}_s & ... & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_s \end{pmatrix} , \tag{6.3.10}$$

where the blocks $\mathbb{D}$ and $\mathbb{B}_i$, $i = 2, ..., m$, were defined in (6.2.59). Also, by Lemma 6.2.3, when $R^\flat$ and $\mathcal{J}_{s,\gamma}$ are fixed, the restriction of the function $\mathtt{Q}$ to the set $\Upsilon^1(\mathcal{P}^\sharp_{\mathtt{a}} \oplus \{R^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$ is affine. Hence, $\mathtt{A}^\sharp_{R^\flat,\mathcal{J}_{s,\gamma}}$ is represented by matrix (6.3.10) acting on $\Upsilon^1(\mathcal{P}^\sharp_{\mathtt{a}} \oplus \{R^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$ plus a constant term depending only on $\mathtt{R}^\flat_{\mathtt{a}} \equiv R^\flat \circ \mathcal{L}^{-1}_{\mathtt{a}}$ and $\mathtt{J}_{s,\gamma,\mathtt{a}} \equiv \mathcal{L}_{\mathtt{a}} \circ \mathcal{J}_{s,\gamma}$; thus, it is globally invertible in $\Upsilon^1(\mathcal{P}^\sharp_{\mathtt{a}} \oplus \{R^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$. We set

$$S^\sharp := \left(\Upsilon^1\right)^{-1} \left(\mathcal{N}\left(\mathtt{Q}^\sharp_{R,\mathcal{J}_{s,\gamma}}\right)\right) \in \mathcal{P}^\sharp_{\mathtt{a}} ,$$

which is equivalent to saying that

$$(S^\sharp + R^\flat, \mathcal{J}_{s,\gamma}) \in \mathcal{N}(\mathtt{Q} \circ \Upsilon^1) ,$$

i.e., by (6.3.8) and (6.2.33), $S^\sharp + R^\flat$ satisfies the s-vanishing condition on $\mathcal{J}_{s,\gamma}$. □

We are now able to prove Lemma 6.3.1.

Proof. (Lemma 6.3.1)

We consider a polynomial $P \in \mathtt{D} \subset \mathcal{P}(r, m) \backslash \Sigma^1(r, s, m)$, with $\overline{\mathtt{D}} \cap \Sigma^1(r, s, m) = \varnothing$.

Case $m = 1$. In case $m = 1$, by Lemma 6.2.1 there exists a constant $C_1(\mathtt{D})$ such that the truncation at order $s + 1$ of P - indicated by P_{s+1} - satisfies

$$||P_{s+1}||_\infty > C_1(\mathtt{D}) .$$

The thesis follows easily by the expression above and by Definition 6.1.3.

Case $m \geq 2$. For any fixed arc $\gamma \in \Theta^1_m$, we shift to its associated adapted coordinates, and we consider the s-truncation $\mathtt{J}_{s,\gamma,\mathtt{a}} \in \vartheta^1_m(s)$, together with the pull-back $\mathtt{P}_{\mathtt{a}} := P \circ \mathcal{L}^{-1}_{\mathtt{a}}$ of the polynomial P w.r.t. the change of coordinates $\mathcal{L}_{\mathtt{a}}$ introduced in paragraph 6.2.3. By the hypothesis, by Lemma 6.2.2 (see especially formulas (6.2.29) and (6.2.31)), and by the fact that the linear terms of the curve γ are uniformly bounded (see the Bernstein's estimate (5.0.1)), there exists a constant $C_1(\mathtt{D}) > 0$ such that

$$||\mathtt{P}_\mathtt{a} - \Sigma^1(r,s,m)||_\infty := \inf_{\widehat{\mathtt{P}}_\mathtt{a} \in \Sigma^1(r,s,m)} ||\mathtt{P}_\mathtt{a} - \widehat{\mathtt{P}}_\mathtt{a}||_\infty > C_1(\mathtt{D}) > 0 \,. \tag{6.3.11}$$

It suffices to prove the statement for the quantity

$$\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial \mathtt{P}_\mathtt{a}}{\partial \mathtt{y}_\ell}\Big|_{\mathtt{J}_{s,\gamma,\mathtt{a}}(t)} \right)_{t=0} \right|$$

instead of

$$\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial \mathtt{P}_\mathtt{a}}{\partial \mathtt{y}_\ell}\Big|_{\mathcal{L}_\mathtt{a}\circ\gamma(t)} \right)_{t=0} \right| \tag{6.3.12}$$

because - as we had already pointed out in paragraph 6.2.4 - the terms of order higher than s in the Taylor developement of $\mathcal{L}_\mathtt{a}\circ\gamma$ yield a null contribution to (6.3.12). With the decomposition in (6.3.7), by Lemmata 6.2.2 and 6.3.2 there exists a unique polynomial $\mathtt{S}_\mathtt{a} = \mathtt{S}^\sharp_\mathtt{a} + \mathtt{S}^\flat_\mathtt{a}$ fulfilling the s-vanishing condition on $\mathtt{J}_{s,\gamma,\mathtt{a}}$ and satisfying $\mathtt{S}^\flat_\mathtt{a} = \mathtt{P}^\flat_\mathtt{a}$. Hence, (6.3.11) yields

$$||\mathtt{P}_\mathtt{a} - \mathtt{S}_\mathtt{a}||_\infty = ||\mathtt{P}^\sharp_\mathtt{a} - \mathtt{S}^\sharp_\mathtt{a}||_\infty > C_1(\mathtt{D}) > 0 \,. \tag{6.3.13}$$

By the proof of Lemma 6.3.2, we also know that - since $\mathtt{S}^\flat_\mathtt{a} = \mathtt{P}^\flat_\mathtt{a}$ and $\mathtt{J}_{s,\gamma,\mathtt{a}}$ are kept fixed - the function $\mathtt{A}^\sharp_{P^\flat,\mathcal{J}_{s,\gamma}}$ in (6.3.9) is affine and invertible in $\Upsilon^1(\mathcal{P}^\sharp_\mathtt{a} \oplus \{P^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$. In particular, it is represented by matrix $\mathcal{A}$ in (6.3.10) plus a constant term depending only on $\mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}$. Taking into account the fact that $\mathtt{A}^\sharp_{P^\flat,\mathcal{J}_{s,\gamma}}$ is the restriction of $\mathtt{Q}$ to the set $\Upsilon^1(\mathcal{P}^\sharp_\mathtt{a} \oplus \{P^\flat\} \times \{\mathcal{J}_{s,\gamma}\})$, one can write

$$||\mathtt{P}^\sharp_\mathtt{a} - \mathtt{S}^\sharp_\mathtt{a}||_\infty \le ||\mathcal{A}^{-1}||_\infty ||\mathtt{Q}(\mathtt{P}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) - \mathtt{Q}(\mathtt{S}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})||_\infty \,, \tag{6.3.14}$$

where $||\mathcal{A}^{-1}||_\infty$ indicates the matrix norm of the inverse. Expressions (6.3.13) and (6.3.14) together yield

$$||\mathtt{Q}(\mathtt{P}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) - \mathtt{Q}(\mathtt{S}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})||_\infty > \frac{C_1(\mathtt{D})}{||\mathcal{A}^{-1}||_\infty} \,. \tag{6.3.15}$$

Moreover, by construction one has $\mathtt{S}^\sharp_\mathtt{a} \in \mathcal{N}(\mathtt{A}^\sharp_{P^\flat,\mathcal{J}_{s,\gamma}})$, that is $(\mathtt{S}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) \in \mathcal{N}(\mathtt{Q})$, so that (6.3.15) implies

$$||\mathtt{Q}(\mathtt{P}^\sharp_\mathtt{a} + \mathtt{P}^\flat_\mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})||_\infty > \frac{C_1(\mathtt{D})}{||\mathcal{A}^{-1}||_\infty} \,. \tag{6.3.16}$$

From the explicit form (6.3.10) of the $(ms+m)\times(ms+m)$ matrix $\mathcal{A}$, one can easily infer the form of $\mathcal{A}^{-1}$, namely

$$\mathcal{A}^{-1} := \begin{pmatrix} \mathbb{I}_m & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathbb{D}^{-1} & -\mathbb{D}^{-1}\mathbb{B}_2 & -\mathbb{D}^{-1}\mathbb{B}_3 & \dots & -\mathbb{D}^{-1}\mathbb{B}_m \\ 0 & 0 & \mathbb{I}_s & 0 & \dots & 0 \\ 0 & 0 & 0 & \mathbb{I}_s & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_s \end{pmatrix}. \tag{6.3.17}$$

The induced matrix norm is, by construction, $||\mathcal{A}^{-1}||_\infty := \sup_{i\in\{1,\dots,n\}} \sum_{j=1}^{ms+m} |\mathcal{A}^{-1}_{ij}|$. By the definition of $\mathbb{D}$ given above table (6.2.60), and by the definition of the blocks $\mathbb{B}_i$, with $i \in \{2,\dots,m\}$, in (6.2.59) one has $||\mathbb{D}^{-1}||_\infty = 1$ and for $m \in \{2,\dots,m\}$ one can write

$$\sup_{\ell\in\{2,\dots,m\}} ||\mathbb{B}_\ell|| \leq \begin{cases} 0 & \text{if } s = 1 \\ s\times(s-1)\max_{\substack{\ell=2,\dots,m \\ \alpha=2,\dots,s}} |a_{\ell\alpha}| & \text{if } 2 \leq s \leq r-1 \end{cases}.$$

Hence, one finally has

$$||\mathcal{A}^{-1}||_\infty \leq \begin{cases} 1 & \text{if } s = 1 \\ 1+(m-1)\,s\,(s-1)\times\max_{\substack{\ell=2,\dots,m \\ \alpha=2,\dots,s}} |a_{\ell\alpha}| & \text{if } 2 \leq s \leq r-1 \end{cases}. \tag{6.3.18}$$

Estimate (6.3.18), together with formulas (6.3.16), (6.3.8), and (6.2.33) implies the thesis, with $C_2(s,m) = (m-1)\,s\,(s-1)$ □

Chapter 7

Proof of Theorem A

In order to prove Theorem A, we need to combine the Lemmata of the previous sections with several intermediate results.

7.1 Codimension estimates

For any pair of integers $n \geq 2$ and $1 \leq m \leq n-1$, we indicate by $O(n,m)$ the space of $n \times m$ real matrices whose columns are orthonormal vectors of $\mathbb{R}^n$. Clearly, a matrix $A \in O(n,m)$ induces a map from $\mathbb{R}^m$ to $\mathbb{R}^n$ associating $\mathbb{R}^m \ni x \longmapsto I = Ax \in \mathbb{R}^n$. From a geometric point of view, for any integer $r \geq 2$, the restriction of any polynomial $Q(I) \in \mathcal{P}(r,n)$ to any m-dimensional subspace $\Gamma^m \subset \mathbb{R}^n$ endowed with the Euclidean metric can be identified through $Q|_{\Gamma^m}(x) := Q(Ax) =: P(x) \in \mathcal{P}(r,m)$, where the columns of $A \in O(n,m)$ span Γ^m.

We also indicate by $O(m)$ the $m \times m$ orthogonal group and by $\mathtt{G}(m,n)$ the m-dimensional Grassmannian, which is locally isomorphic to $O(n,m)/O(m)$.

With this setting, for any integer $1 \leq s \leq r-1$, we define

$$\begin{aligned} \mathcal{U} = \mathcal{U}(r,s,m,n) := \{(Q,A,P) \in \mathcal{P}^\star(r,n) \times O(n,m) \times \mathcal{P}(r,m)| \\ P(x) = Q(Ax)\ , \quad P(x) \in \Sigma(r,s,m)\}\ , \end{aligned} \tag{7.1.1}$$

and we indicate by

$$\mathcal{V} = \mathcal{V}(r,s,m,n) := \Pi_{\mathcal{P}^\star(r,n)}\, \mathcal{U}(r,s,m,n) \tag{7.1.2}$$

its projection onto the first component, i.e. the set of those polynomials $Q \in \mathcal{P}^\star(r,n)$ for which the origin is non-critical, and such that, for some m-dimensional subspace Γ^m orthogonal to $\nabla Q(0)$, the restriction $Q|_{\Gamma^m} \in \mathcal{P}(r,m)$ belongs to the closure $\Sigma(r,s,m)$ of the set of s-vanishing polynomials introduced in section 6.

Remark 7.1.1. We observe that it is implicit in Definition (7.1.1) that Γ^m must be orthogonal to $\nabla Q(0)$. Infact, any $P \in \sigma(r,s,m)$ must satisfy $\nabla P(0) = 0$ (see equation (6.2.4)).

Hence, the limit $\widehat{P}$ of any converging sequence $\{P_n \in \sigma(r,s,m)\}_{n\in\mathbb{N}}$ must have the same property. Since $\Sigma(r,s,m) = \overline{\sigma}(r,s,m)$, one has $\nabla\widehat{P}(0) = 0$ for any $\widehat{P} \in \Sigma(r,s,m)$. As in our case we are considering $\widehat{P}(x) = \widehat{Q}(Ax)$ for some $\widehat{Q} \in \mathcal{P}^\star(r,n)$, we have $\nabla\widehat{P}(0) = A^\dagger\nabla\widehat{Q}(0) = 0$, which is equivalent to saying that the columns of A are all orthogonal to $\nabla\widehat{Q}(0)$.

The first result that we prove in this section is the following

Lemma 7.1.1. *$\mathcal{V}(r,s,m,n)$ is a closed set in $\mathcal{P}^\star(r,n)$ for the topology induced by $\mathcal{P}(r,n)$.*

Proof. Consider a sequence $\{Q_j\}_{j\in\mathbb{N}}$ in $\mathcal{V}(r,s,m,n)$, converging to some polynomial $\overline{Q} \in \mathcal{P}^\star(r,n)$ for the topology induced by $\mathcal{P}(r,n)$ on $\mathcal{P}^\star(r,n)$. Then, for any fixed $j \in \mathbb{N}$, by (7.1.1)-(7.1.2) there exists $A_j \in O(n,m)$ such that $P_j(x) := Q_j(A_jx) \in \Sigma(r,s,m)$. Since $O(n,m)$ is compact, there exists $\overline{A} \in O(n,m)$ and a subsequence $\{A_{j_k}\}_{k\in\mathbb{N}} \longrightarrow \overline{A}$. Hence, there exists a polynomial $\overline{P} \in \mathcal{P}(r,m)$ such that the subsequence $\{P_{j_k}(x) := Q_{j_k}(A_{j_k}x)\}_{k\in\mathbb{N}}$ converges to $\overline{P}(x) := \overline{Q}(\overline{A}x)$. Since $P_{j_k}(x) \in \Sigma(r,s,m)$ for all $k \in \mathbb{N}$ by construction, and $\Sigma(r,s,m)$ is closed, $\overline{P}(\overline{A}x) \in \Sigma(r,s,m)$, whence the thesis. □

Moreover, for given values of m,n, when r and $1 \le s \le r-1$ are sufficiently high, the set $\mathcal{V}(r,s,m,n)$ becomes generic, namely

Lemma 7.1.2. *$\mathcal{V}(r,s,m,n)$ is a semi-algebraic set of $\mathcal{P}^\star(r,n)$ satisfying*

$$\textit{codim}\ \mathcal{V}(r,s,m,n) \ge \max\{0, s - m(n-m-1)\}\ . \tag{7.1.3}$$

Proof. By hypothesis, $\Sigma(r,s,m) := \overline{\sigma}(r,s,m)$ and $\sigma(r,s,m)$ is a semi-algebraic set (see Corollary 6.2.1). Hence, Proposition A.1.2 assures that $\Sigma(r,s,m)$ is also semi-algebraic. Therefore, set $\mathcal{U}$ in (7.1.1) is clearly semi-algebraic, since it is determined by a finite number of semi-algebraic relations. Finally, the Theorem of Tarski and Seidenberg A.1.1 implies that $\mathcal{V}$ is semi-algebraic since it is obtained by projecting $\mathcal{U}$ onto its first component.

As for the codimension of $\mathcal{V}$, we start by estimating the dimension of $\mathcal{U}$. We remark that, for a fixed choice of $\overline{A} \in O(n,m)$ and $\overline{P} \in \Sigma(r,s,m)$, one has

$$\begin{aligned}&\dim(\mathcal{U} \cap \{(Q,A,P) \in \mathcal{P}^\star(r,n)\times O(n,m)\times\mathcal{P}(r,m)\ :\ A = \overline{A}\ ,\quad P = \overline{P}\})\\ &= \dim\mathcal{P}^\star(r,n) - \dim\mathcal{P}(r,m)\ .\end{aligned} \tag{7.1.4}$$

Infact, since the matrix $A := (A_1|\dots|A_m)$, $A_1,\dots,A_m \in \mathbb{R}^n$, has been fixed, one can construct an orthonormal basis of $\mathbb{R}^n$ by completing $A_1,\dots,A_m$, with $n-m$ vectors $A_{m+1},\dots,A_n$. Since in the set above the restriction of any polynomial $Q \in \mathcal{P}(r,n)$ to the space generated by $A_1,\dots,A_m$ is fixed, all the monomials of Q corresponding to the coordinates associated to $A_1,\dots,A_m$ are uniquely determined. The number of these monomials is $\dim\mathcal{P}(r,m)$, and the total number of monomials in Q in $\dim\mathcal{P}(r,n)$, whence equality (7.1.4). In order to compute $\dim \mathcal{U}$, one must add to the r.h.s. of (7.1.4) the dimension of the spaces corresponding to the fixed variables, namely

$$\begin{aligned}\dim \mathcal{U} &= \dim(\mathcal{U} \cap \{(Q, A, P) \in \mathcal{P}^\star(r,n) \times O(n,m) \times \mathcal{P}(r,m) \;:\; A = \overline{A}\,, \quad P = \overline{P}\}) \\ &\quad + \dim O(n,m) + \dim \Sigma(r,s,m) \\ &= \dim \mathcal{P}^\star(r,n) - \dim \mathcal{P}(r,m) + \dim O(n,m) + \dim \Sigma(r,s,m)\;. \end{aligned} \tag{7.1.5}$$

We observe that, by Definition 7.1.1, if $(Q(I), A, Q(Ax)) \in \mathcal{U}$, for any orthogonal matrix $M \in O(m)$ also $(Q(I), AM, Q(AMx)) \in \mathcal{U}$, since one has the freedom to choose the orthonormal basis spanning the m-dimensional subspace $\Gamma^m \in \mathtt{G}(m,n)$. More precisely, if we define the action of $O(m)$ on any element $(Q(I), A, Q(Ax)) \in \mathcal{U}$ as

$$(Q(I), A, Q(Ax)) \longmapsto (Q(I), AM, Q(AMx))\;, \tag{7.1.6}$$

we can define an orbit of $O(m)$ starting at a given point $(Q(I), A, Q(Ax)) \in \mathcal{U}$ as

$$\{(Q(I), AM, Q(AMx)) \in \mathcal{U}\;, \quad M \in O(m)\}\;. \tag{7.1.7}$$

Since the first component in (7.1.7) is invariant, by (7.1.2) we see that the set $\mathcal{V}$ can be found as the projection of the set of orbits $\mathcal{U}/O(m)$ onto $\mathcal{P}^\star(r,n)$, namely

$$\mathcal{V} := \Pi_{\mathcal{P}^\star(r,n)}\,\mathcal{U} = \Pi_{\mathcal{P}^\star(r,n)}\,(\mathcal{U}/O(m))\;, \tag{7.1.8}$$

so that one can write

$$\dim \mathcal{V} = \dim \Pi_{\mathcal{P}^\star(r,n)}\,(\mathcal{U}/O(m))\;. \tag{7.1.9}$$

Obviously, the action of $O(m)$ on $\mathcal{U}$ is free and smooth, hence by C.1.1 it is also proper. Therefore, Theorem C.1.1 assures that

$$\dim(\mathcal{U}/O(m)) = \dim \mathcal{U} - \dim O(m)\;. \tag{7.1.10}$$

By (7.1.9), we have

$$\operatorname{codim} \mathcal{V} \geq \; \operatorname{codim}\,(\mathcal{U}/O(m)) - \dim O(n,m) - \dim \mathcal{P}(r,m) \tag{7.1.11}$$

and equations (7.1.5) and (7.1.10) imply

$$\begin{aligned}&\operatorname{codim}\,(\mathcal{U}/O(m)) \\ &= \dim \mathcal{P}^\star(r,n) + \dim O(n,m) + \dim \mathcal{P}(r,m) - \dim \mathcal{U} + \dim O(m)\;. \\ &\geq 2\dim \mathcal{P}(r,m) - \dim \Sigma(r,s,m) + \dim O(m)\end{aligned} \tag{7.1.12}$$

Expressions (7.1.11) and (7.1.12) together yield

$$\begin{aligned}\operatorname{codim} \mathcal{V} &\geq \dim \mathcal{P}(r,m) - \dim \Sigma(r,s,m) + \dim O(m) - \dim O(n,m) \\ &= \; \operatorname{codim} \Sigma(r,s,m) + \dim O(m) - \dim O(n,m)\;.\end{aligned} \tag{7.1.13}$$

By Proposition A.1.3, $\operatorname{codim}\Sigma(r,s,m) = \operatorname{codim}\sigma(r,s,m)$, so by Corollary 6.2.1 one has $\operatorname{codim}\Sigma(r,s,m) = s+m$. Moreover, since $\dim O(m) = m(m-1)/2$ and $\dim O(n,m) = mn - m(m-1)/2 - m$, (7.1.13) reads

$$\operatorname{codim} \mathcal{V} \geq s - m(n-m-1)\,. \tag{7.1.14}$$

Since the codimension is a nonnegative number, the thesis follows.

□

7.2 Stable lower estimates

The set $\mathcal{V}(r,s,m,n)$ introduced in the previous section is important because, for any given polynomial $Q \in \mathcal{P}^\star(r',n)$ - with $r' \geq r$ - whose truncation at order r lies outside of $\mathcal{V}(r,s,m,n)$, all polynomials belonging to a small open neighborhood of Q in $\mathcal{P}^\star(r',n)$ are steep around the origin on the subspaces of dimension m, with uniform indices and coefficients. More precisely, one has

Theorem 7.2.1. *Take five integers $r' \geq r \geq 2$, $1 \leq s \leq r-1$, $n \geq 2$, $m \in \{1,...,n-1\}$. Consider a polynomial $Q \in \mathcal{P}^\star(r',n)$, and suppose that for some $\tau > 0$ its truncation at order r, indicated by Q_r, satisfies*

$$\big|\big|Q_r - \mathcal{V}(r,s,m,n)\big|\big|_\infty := \inf_{R\in\mathcal{V}(r,s,m,n)} ||Q_r - R||_\infty > \tau\,. \tag{7.2.1}$$

There exist constants $\varepsilon_0 = \varepsilon_0(r,s,m,\tau,n)$, $\mathtt{C}_m = \mathtt{C}_m(r',r,s,\tau,n)$, $\lambda_0 = \lambda_0(r,s,m,\tau)$, and $\widehat{\delta} = \widehat{\delta}(r,s,m,\tau,n)$ such that any polynomial $S \in \mathcal{P}^\star(r',n)$ contained in a ball of radius $\varepsilon \in [0,\varepsilon_0]$ around Q in $\mathcal{P}^\star(r',n)$ is steep on the subspaces of dimension m at any point $I \in B^n(0,\widehat{\delta})$, with uniform steepness coefficients $\mathtt{C}_m$, λ_0 and with steepness indices bounded by

$$\overline{\alpha}_m(s) := \begin{cases} s \quad , & \text{if } m = 1 \\ 2s-1 \quad , & \text{if } m \geq 2\,. \end{cases} \tag{7.2.2}$$

Remark 7.2.1. By Lemma 7.1.2, since $1 \leq s \leq r-1$, if $r > m(n-m-1)+1$ and $s \geq m(n-m-1)+1$ one has $\operatorname{codim}\mathcal{V}(r,s,m,n) \geq 1$, so that hypothesis (7.2.1) is generic in $\mathcal{P}^\star(r,n)$.

In order to prove Theorem 7.2.1, we need a Lemma used by Pyartli in the study Diophantine approximation. This result is crucial in KAM Theory, as it was shown by Herman, Rüssmann, Sevryuk, and others (see [39] and the references therein). We give its statement in a version provided by Rüssmann [108].

Lemma 7.2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$, with $a < b$, be a q-times continuously differentiable function satisfying*

$$|f^{(q)}(t)| > \beta$$

for all $t \in [a, b]$, *for some* $q \in \mathbb{N}$ *and* $\beta > 0$.

Then one has the estimate

$$\text{meas}\{t \in [a, b] \mid |f(t)| \leq \rho\} \leq 4\left(q! \frac{\rho}{2\beta}\right)^{1/q}$$

for all $\rho > 0$.

We also need the following auxiliary

Lemma 7.2.2. *With the hypotheses of Theorem 7.2.1, there exist positive constants* $\varepsilon^\star = \varepsilon^\star(r, s, m, \tau, n)$, $\chi = \chi(r, s, m, \tau, n)$, *and* $\zeta = \zeta(r, s, m, \tau)$, *such that, for any* $\varepsilon \in [0, \varepsilon^\star]$, *the truncation* $S_r \in \mathcal{P}^\star(r, n)$ *of any polynomial* $S \in \mathcal{P}^\star(r', n)$ *contained in a ball of radius* ε *around* Q *in* $\mathcal{P}^\star(r', n)$ *verifies*

$$||S_r - \mathcal{V}(r, s, m, n)||_\infty > \chi \;, \tag{7.2.3}$$

and for any m*-dimensional subspace* Γ^m *orthogonal to* $\nabla S(0)$, *one has*

$$||S_r|_{\Gamma^m} - \Sigma(r, s, m)||_\infty > \zeta \;. \tag{7.2.4}$$

Proof. (Lemma 7.2.2)

We split the proof into three steps. In the first one, we introduce appropriate sets and notations that are helpful in the proof. In second step, we introduce suitable continuous functions by exploiting the existence of local continuous sections for the Grassmannian. In the third and last step, the statement is proved by exploiting the first two steps and the compactness of the Grassmannian.

We also observe that, due to Remark 7.2.1, estimate (7.2.3) is trivial for $r > m(n - m - 1) + 1$ and $s \geq m(n - m - 1) + 1$.

Step 1. For any given polynomial $V \in \mathcal{P}^\star(r, n)$, we denote by $\mathtt{G}_V(m, n) \subset \mathtt{G}(m, n)$ the compact subset of m-dimensional subspaces orthogonal to $\nabla V(0) \neq 0$. We also define the set

$$\Lambda^m := \{(V, \Gamma^m) \mid V \in \mathcal{P}^\star(r, n),\ \Gamma^m \in \mathtt{G}_V(m, n)\} \;. \tag{7.2.5}$$

Now, setting $N := \dim \mathcal{P}^\star(r', n)$, for sufficiently small ε one has $\nabla S(0) \neq 0$ for any $S \in B^N(Q, \varepsilon)$. Moreover, the map $\mathtt{f} : \mathcal{P}^\star(r', n) \longrightarrow \mathbb{R}^n$ associating $S \longmapsto \nabla S(0)$ is obviously continuous and surjective, and the same holds true for the function $\mathtt{h} : \mathbb{R}^n \longrightarrow \mathtt{G}(n-1, n)$ which to a vector ω associates $\omega^\perp$. Hence, $\mathtt{h}\circ\mathtt{f}$ is also continuous and surjective. Therefore there exists an open set of $n - 1$ dimensional hyperplanes - indicated by $\mathcal{W}^{n-1}(Q, \varepsilon) \subset \mathtt{G}(n-1, n)$ - whose inverse image $\mathtt{f}^{-1}(\mathtt{h}^{-1}(\mathcal{W}^{n-1}(Q, \varepsilon)))$ contains $B^N(Q, \varepsilon)$. Hence, for $m \in \{1, \dots, n-1\}$, we can define the open set

$$\mathcal{W}^m(Q, \varepsilon) := \{\Gamma^m \in \mathtt{G}(m, n) \mid \Gamma^m \subseteq \Gamma^{n-1} \text{ for some } \Gamma^{n-1} \in \mathcal{W}^{n-1}(Q, \varepsilon)\} \;.$$

The above construction implies that, for any $m \in \{1, ..., n-1\}$ and for sufficiently small ε, the choice of an open ball $B^N(Q, \varepsilon)$ determines a set

$$\Xi^m(Q, \varepsilon) := \{(S, \Gamma^m) \mid S \in B^N(Q, \varepsilon), \Gamma^m \in \mathcal{W}^m(Q, \varepsilon), \Gamma^m \in \mathtt{G}_S(m, n)\} \subset \Lambda^m \;. \tag{7.2.6}$$

Remark 7.2.2. To carry out the construction at this step, one only needs to perturb the linear terms of Q. The bound on ε that must be considered at this step, therefore, does not depend on the degree of the polynomial Q.

Step 2. Now, we take into account the fact that it is always possible to define a local continuous section for the Grassmannian $\mathtt{G}(m,n)$. Namely, for any element $\Gamma \in \mathtt{G}(m,n)$ there exists an open neighborhood $\mathcal{E}_\Gamma \subset \mathtt{G}(m,n)$ of Γ and a continuous map $\xi : \mathcal{E}_\Gamma \longrightarrow O(n,m)$ such that, if $\pi : O(n,m) \longrightarrow \mathtt{G}(m,n)$ is the canonical projection, then $\pi\circ\xi$ is the identity. Hence, for any m-dimensional subspace $\Gamma^m \in \mathtt{G}_Q(m,n)$, we consider its associated open neighborhood $\mathcal{E}_{\Gamma^m}$, and a compact neighborhood $\overline{\mathtt{V}}_{\Gamma^m} \subset \mathcal{E}_{\Gamma^m}$ centered at Γ^m. Since $\mathtt{G}_Q(m,n)$ is compact, it can be covered by a finite number $L > 0$ of compact neighborhoods $\overline{\mathtt{V}}_i$ and open neighborhoods $\mathcal{E}_i$, with $i = 1, ..., L$, of this kind. Hence, if ε is sufficiently small, then one has

$$\mathcal{W}^m(Q,\varepsilon) \subset \bigcup_{i=1}^{L} \overline{\mathtt{V}}_i \subset \bigcup_{i=1}^{L} \mathcal{E}_i \; . \tag{7.2.7}$$

Moreover, if we indicate by ξ_i, $i = 1, ..., L$ the continuous section associated to each neighborhood $\mathcal{E}_i$, it makes sense to define the sets

$$\Xi_i^m(Q,\varepsilon) := \{(S,\Gamma^m) \,|\, (S,\Gamma^m) \in \Xi^m(Q,\varepsilon), \Gamma^m \in \mathcal{E}_i\} \; , \quad \Xi^m(Q,\varepsilon) = \bigcup_{i=1}^{L} \Xi_i^m(Q,\varepsilon) \tag{7.2.8}$$

and the continuous functions

$$\begin{aligned} F_i \,:\, & \Xi_i^m(Q,\varepsilon) \longrightarrow \mathcal{P}(r',m) \\ & (S(I),\Gamma^m) \longmapsto T(x) := S(Ax) \; , \quad A := \xi_i(\Gamma^m) \in O(n,m) \; . \end{aligned} \tag{7.2.9}$$

Step 3. Fix $i = 1, ..., L$. By hypothesis $||Q_r - \mathcal{V}||_\infty > \tau$ and $\nabla Q(0) = \nabla Q_r(0) \neq 0$, so that by the definition of $\mathcal{U}$ and $\mathcal{V}$ in (7.1.1)-(7.1.2), by Remark 7.1.1, and by the compactness of $\mathtt{G}_Q(m,n) = \mathtt{G}_{Q_r}(m,n)$, there exists $\zeta_i = \zeta_i(r,s,m,\tau) > 0$ such that - on any subspace $\Gamma^m \in \mathtt{G}_Q(m,n)$, $\Gamma^m \in \mathcal{E}_i$ - one has

$$||P_r - \Sigma(r,s,m)||_\infty > 2\,\zeta_i \; , \tag{7.2.10}$$

where $P_r(x) := Q_r(Ax)$ - with $A = \xi_i(\Gamma^m)$ - is the restriction to the subspace Γ^m of the truncation Q_r.

Now, fix $\Gamma^m \in (\mathtt{G}_Q(m,n) \cap \overline{\mathtt{V}}_i) \subset \mathcal{E}_i$. By the continuity of F_i, there exists $\varepsilon^\star_{i,\Gamma^m} = \varepsilon^\star_{i,\Gamma^m}(r,s,m,\tau,n) > 0$ such that, for any $\varepsilon \in]0, \varepsilon^\star_{i,\Gamma^m}]$, the open ball $v^m_{i,\Gamma^m}(\varepsilon) \subset \Xi_i^m(Q,\varepsilon)$ centered at (Q,Γ^m) verifies the following property: for any $(S,\widehat{\Gamma}^m) \in v^m_{i,\Gamma^m}(\varepsilon)$, the restricted truncated polynomial $\widehat{T}_r(x) := S_r(\widehat{A}x)$, with $\widehat{A} = \xi_i(\widehat{\Gamma}^m)$, is contained in an open ball of radius ζ_i around $P_r(x) := Q_r(Ax)$, with $A = \xi_i(\Gamma^m)$. Hence, on the one hand by (7.2.10) one infers

$$||\widehat{T}_r - P_r||_\infty < \zeta_i \quad \Longrightarrow \quad ||\widehat{T}_r - \Sigma(r,s,m)||_\infty > \zeta_i \; . \tag{7.2.11}$$

On the other hand, by construction one has

$$\left(Q, \mathtt{G}_Q(m,n) \cap \overline{\mathtt{V}}_i\right) \subset \bigcup_{\Gamma^m \in \mathtt{G}_Q(m,n)\cap\overline{\mathtt{V}}_i} v^m_{i,\Gamma^m}(\varepsilon^\star_{i,\Gamma^m}) \tag{7.2.12}$$

and - due to the compactness of the fiber $(Q, \mathtt{G}_Q(m,n) \cap \overline{\mathtt{V}}_i)$ - it is possible to extract a finite number J_i of subspaces $\Gamma^m \in \mathtt{G}_Q(m,n) \cap \overline{\mathtt{V}}_i$ from (7.2.12) and write (with slight abuse of notation)

$$\left(Q, \mathtt{G}_Q(m,n) \cap \overline{\mathtt{V}}_i\right) \subset \bigcup_{j=1}^{J_i} v^m_{i,j}(\varepsilon) \subset \Xi^m_i(Q,\varepsilon) \quad , \quad \varepsilon \in \left]0, \varepsilon^\star_i\right] \ , \tag{7.2.13}$$

where we have set

$$\varepsilon^\star_i = \varepsilon^\star_i(r,s,m,\tau,n) := \min_{j\in\{1,\ldots,J_i\}} \{\varepsilon^\star_{i,j}\} \ . \tag{7.2.14}$$

Inclusion (7.2.13), together with (7.2.8), yields that the finite union $v^m(\varepsilon) := \cup_{i=1}^{L} \cup_{j=1}^{J_i} v^m_{i,j}(\varepsilon)$ is an open neighborhood of $\Xi^m(Q,\varepsilon)$ containing the fiber $(Q, \mathtt{G}_Q(m,n))$. Therefore, by setting

$$\zeta = \zeta(r,s,m,\tau) := \min_{i\in\{1,\ldots,L\}} \{\zeta_i\} > 0 \ , \quad \varepsilon^\star = \varepsilon^\star(r,s,m,\tau,n) := \min_{i=1,\ldots,L} \{\varepsilon^\star_i\} > 0 \ ,$$

and by taking (7.2.11) into account, one has that for $0 < \varepsilon \leq \varepsilon^\star$ and for any $(S, \widehat{\Gamma}^m) \in v^m(\varepsilon) \subset \Xi^m(Q,\varepsilon)$, the restricted truncated polynomial $\widehat{T}_r(x) := S_r(Ax)$ - with $A = \xi_j(\widehat{\Gamma}^m)$ for some $j = 1, ..., L$ - verifies

$$||\widehat{T}_r - \Sigma(r,s,m)||_\infty > \zeta_j \geq \zeta \ . \tag{7.2.15}$$

Therefore, we have proved that, for any $0 \leq \varepsilon \leq \varepsilon^\star$ there exists $\zeta > 0$ such that

$$||S_r|_{\Gamma^m} - \Sigma(r,s,m)||_\infty > \zeta \ , \tag{7.2.16}$$

for any $S \in B^N(Q,\varepsilon)$ and for any Γ^m orthogonal to $\nabla S(0) = \nabla S_r(0) \neq 0$. Hence, the Definition of set $\mathcal{V}(r,s,m,n)$ in (7.1.2) ensures that for any $0 \leq \varepsilon \leq \varepsilon^\star$ there exists $\chi = \chi(r,s,m,\tau,n) > 0$ such that for any $S \in B^N(Q,\varepsilon)$ one has

$$||S_r - \mathcal{V}(r,s,m,n)||_\infty > \chi \ . \tag{7.2.17}$$

This concludes the proof.

□

We need another intermediate result in order to demonstrate Theorem 7.2.1. Before giving its statement, for any polynomial $S \in \mathcal{P}^\star(r', n)$, we firstly consider its associated minimal arc γ constructed in Theorem 5.0.1. Also, for any $\lambda > 0$ we indicate by $\mathtt{I}'_\lambda \subset$

$[-\lambda, \lambda]$ the interval obtained by cutting the interval $\mathtt{I}_\lambda$ at point 3) of Theorem 5.0.1 into three equal pieces and by taking the central one. In particular, we have

$$|\mathtt{I}'_\lambda| = \frac{|\mathtt{I}_\lambda|}{3} = \frac{\lambda}{3\mathtt{K}} ,$$

where $\mathtt{K} = \mathtt{K}(r', m, n)$ is a suitable constant.

We also assume the setting of Lemmata 7.2.1 and 7.2.2, and we consider a ball $B^N(Q, \varepsilon)$ of radius $\varepsilon < \varepsilon^\star/2$ around $Q \in \mathcal{P}^\star(r', n)$. Within this framework, one has

Lemma 7.2.3. *There exist two constants $C' = C'(r', r, s, m, \tau)$ and $\lambda_0 = \lambda_0(r, s, m, \tau)$ such that - for any number $0 < \lambda \le \lambda_0$, for any polynomial $S \in \mathcal{P}^\star(r', n)$ contained in $B^N(Q, \varepsilon)$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla S(0)$, the restriction $T := S|_{\Gamma^m}$ satisfies the following estimates*

$$\begin{aligned} &\max_{\alpha=1,\dots,s} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial T(x)}{\partial x_1}\Big|_{\gamma(t)} \right)_{t=t^\star} \right| > C' \quad , \qquad \textit{for } m = 1, \\ &\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial T(x)}{\partial x_\ell}\Big|_{\gamma(t)} \right)_{t=t^\star} \right| > C' \lambda^{s-1} \quad , \qquad \textit{for } m \ge 2 \end{aligned} \tag{7.2.18}$$

at any point $t^\star \in \mathtt{I}'_\lambda$.

Proof. (Lemma 7.2.3) We proceed by steps.

Step 1. We consider the value $\varepsilon^\star$ of Lemma 7.2.2 and we fix a polynomial S in the ball $B^N(Q, \varepsilon)$, where $\varepsilon \in [0, \varepsilon^\star/2]$. By Lemma 7.2.2, there exists a parameter $\zeta = \zeta(r, s, m, \tau) > 0$ such that on any m-dimensional subspace Γ^m orthogonal to $\nabla S(0) \neq 0$ - the truncation S_r varifies

$$||S_r|_{\Gamma^m} - \Sigma(r, s, m)||_\infty > \zeta \ . \tag{7.2.19}$$

Now, for a given subspace $\Gamma^m \in \mathtt{G}_S(m, n)$, one can choose a matrix $A \in O(n, m)$ whose columns span Γ^m and set $T(x) := S(Ax)$. Then, for any $\lambda > 0$, by Theorem 5.0.1 there exists a minimal real-analytic arc

$$\gamma(t) := \begin{cases} x_1(t) = t \\ x_j(t) = f_j(t), \ j \in \{2, ..., m\} \end{cases} \quad t \in \mathtt{I}_\lambda \subset [-\lambda, \lambda] \ , \ \ |\mathtt{I}_\lambda| = \lambda/\mathtt{K}(r', n, m) \ , \tag{7.2.20}$$

whose image is contained in the thalweg $\mathcal{T}(S, \Gamma^m)$. We observe that, up to a change in the order of the vectors spanning Γ^m, in Theorem 5.0.1, we can always suppose that the coordinate parametrizing γ is the first one. We divide the interval $\mathtt{I}_\lambda$ into three equal parts of length $\lambda/(3\mathtt{K})$ and we denote by $\mathtt{I}'_\lambda := [\lambda_{min}, \lambda_{max}]$ the central one. Then, for any given $t^\star \in \mathtt{I}'_\lambda$ associated to the point $\gamma(t^\star) = x^\star$, we consider the affine reparametrization

$$\gamma^\star(u) := \begin{cases} x_1(u) = u + t^\star \\ x_j(u) = f_j(u + t^\star) \ , \quad j \in \{2, ..., m\} \end{cases} \quad u \in [\lambda_{min} - t^\star, \lambda_{max} - t^\star] \ .$$

It is clear that γ and $\gamma^\star$ share the same image, have the same speed everywhere, and that $\gamma^\star(0) = \gamma(t^\star) = x^\star$, so that for all $\alpha \in \{0, ..., s\}$ and $\ell \in \{1, ..., m\}$, one has

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial T(x)}{\partial x_\ell}\Big|_{\gamma(t)} \right)_{t=t^\star} = \frac{d^\alpha}{du^\alpha}\left(\frac{\partial T(x)}{\partial x_\ell}\Big|_{\gamma^\star(u)} \right)_{u=0} . \tag{7.2.21}$$

Now, indicating by $L_\star : \mathbb{R}^m \longrightarrow \mathbb{R}^m, x \longmapsto x - x^\star =: \widetilde{x}$ the translation w.r.t. $x^\star$, the curve $\gamma^\star$ is mapped into $\widetilde{\gamma}^\star := L_\star \circ \gamma^\star$, which reads

$$\widetilde{\gamma}^\star(u) := \begin{cases} \widetilde{x}_1(u) = x_1(u) - x_1^\star = u + t^\star - t^\star = u \\ \widetilde{x}_j(u) := x_j(u) - x_j(t^\star) = f_j(u + t^\star) - f_j(t^\star) =: \widetilde{f}_j(u) \end{cases} , \tag{7.2.22}$$

with $u \in [\lambda_{min} - t^\star, \lambda_{max} - t^\star]$. The polynomial T written in the new coordinates reads

$$T(x) = T \circ L_\star^{-1}(\widetilde{x}) = T(\widetilde{x} + x^\star) =: U^\star(\widetilde{x}) . \tag{7.2.23}$$

Since $x^\star$ is fixed, one has

$$\frac{\partial T(x)}{\partial x_\ell} = \frac{\partial U^\star(\widetilde{x})}{\partial \widetilde{x}_\ell} , \quad \forall \ell \in \{1, ..., m\} . \tag{7.2.24}$$

Moreover, if one takes into account the fact that $\widetilde{\gamma}^\star(0) = 0$, and that the origin for the coordinates $\widetilde{x}$ corresponds to the point $x = x^\star$ in the old coordinates, equality (7.2.21), together with (7.2.24), yields, for all $\alpha \in \{0, ..., s\}$ and for all $\ell \in \{1, ..., m\}$,

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial T(x)}{\partial x_\ell}\Big|_{\gamma(t)} \right)_{t=t^\star} = \frac{d^\alpha}{du^\alpha}\left(\frac{\partial T(x)}{\partial x_\ell}\Big|_{\gamma^\star(u)} \right)_{u=0} = \frac{d^\alpha}{du^\alpha}\left(\frac{\partial U^\star(\widetilde{x})}{\partial \widetilde{x}_\ell}\Big|_{\widetilde{\gamma}^\star(u)} \right)_{u=0} . \tag{7.2.25}$$

Step 2. Bernstein's estimate (5.0.1) applied to the components of γ in (7.2.20) reads

$$\max_{t \in \mathtt{I}'_\lambda} |f_j(t)| \le \mathtt{M}_0 \lambda \quad , \qquad j = 2, \dots, m \tag{7.2.26}$$

for some uniform constant $\mathtt{M}_0 = \mathtt{M}(r', n, m, 0)$, so that - for any $t^\star \in \mathtt{I}'_\lambda$ - one has

$$||x^\star||_\infty := ||\gamma(t^\star)||_\infty \le \mathtt{M}_0 \lambda . \tag{7.2.27}$$

For any given $\varepsilon \in [0, \varepsilon^\star]$, with the help of the arguments in the proof of Lemma 7.2.2 - in particular taking (7.2.6), (7.2.8), and (7.2.9) into account - the set $\Xi^m(Q, \varepsilon)$ admits the covering $\Xi^m(Q, \varepsilon) = \cup_{i=1}^L \Xi_i^m(Q, \varepsilon)$, and for any index $i \in \{1, ..., L\}$ there exists a continuous function

$$F_i : \Xi_i^m(Q, \varepsilon) \longrightarrow \mathcal{P}(r', m) \tag{7.2.28}$$

that maps

$$(R, \Gamma^m) \longmapsto R(Ax) , \quad A := \xi_i(\Gamma^m) \in O(n, m) , \tag{7.2.29}$$

where ξ_i is a local continuous section for the Grassmannian $\mathtt{G}(m, n)$.

Hence, taking Lemma 7.2.2 and formula (7.2.23) into account, the function

$$\begin{aligned} \mathtt{F}_i^\star \, : \, & \Xi_i^m(Q,\varepsilon)\times\mathbb{R}^m \longrightarrow \mathcal{P}(r',m) \\ & (R,\Gamma^m,x^\star) \longmapsto R(A(\widetilde{x}+x^\star))\,, \quad A := \xi_i(\Gamma^m)\in O(n,m) \end{aligned} \tag{7.2.30}$$

is continuous. Moreover, (7.2.27) and the Theorem of Heine-Cantor ensure that $\mathtt{F}_i^\star$ is uniformly continuous over the restricted compact domain $\overline{\Xi}_i^m(Q,\varepsilon^\star/2)\times\overline{B}_\infty^n(0,\mathtt{M}_0\lambda)$. Choosing the value $\zeta(r,s,m,\tau)>0$ in (7.2.19), there exists a uniform positive real $0<\lambda_i=\lambda_i(\zeta)\le 1$ such that, for any $0<\lambda\le\lambda_i$ and for any $(S,\Gamma^m)\in\Xi_i^m(Q,\varepsilon^\star/2)$, the image of the set $(S,\Gamma^m,\overline{B}_\infty^n(0,\mathtt{M}_0\lambda))$ through $\mathtt{F}_i^\star$ verifies

$$\mathtt{F}_i^\star\left(S,\Gamma^m,\overline{B}_\infty^n(0,\mathtt{M}_0\lambda)\right)\subset B_\infty^M\left(\mathtt{F}_i^\star(S,\Gamma^m,0),\frac{\zeta}{2}\right)\,,\quad M:=\dim\mathcal{P}(r',m)\,. \tag{7.2.31}$$

The above reasonings imply for any $t^\star\in\mathtt{I}'_\lambda\subset[-\lambda,\lambda]$, with $0<\lambda\le\lambda_i$, one has

$$||(T\circ L_\star^{-1})_r-T_r||_\infty=:\,||U_r^\star-T_r||_\infty\le||U^\star-T||_\infty<\frac{\zeta}{2}\,, \tag{7.2.32}$$

where $U_r^\star, T_r$ are the truncations at order r of polynomial $U^\star$ introduced in (7.2.23) and of polynomial $T:=S|_{\Gamma^m}$, respectively.

Repeating the same argument for any index $j\in\{1,...,L\}$, relations (7.2.19), (7.2.31), and (7.2.32) imply that, if

$$0<\lambda\le\lambda_0:=\min_{i\in\{1,\dots,L\}}\{\lambda_i=\lambda_i(\xi(r,s,m,\tau))\}\le 1\,, \tag{7.2.33}$$

then, for any $(S_r,\Gamma^m)\in\Xi^m(Q_r,\varepsilon/2)$ and for any $t^\star\in\mathtt{I}'_\lambda$ one has

$$||U_r^\star-\Sigma(r,m,n)||_\infty=||T_r\circ L_\star^{-1}-\Sigma(r,m,n)||_\infty>\frac{\zeta}{2}\,. \tag{7.2.34}$$

Step 3. Without any loss of generality, we suppose that the minimal curve $\gamma\subset\mathcal{T}(S,\Gamma^m)$ is parametrized by the first coordinate. Hence, for $n\ge 3$ and $2\le m\le n-1$ we indicate by $\mathtt{a}=(a_{12},...,a_{1m})$ the linear coefficients of the Taylor expansion of the translated curve $\widetilde{\gamma}^\star$ in (7.2.22). One can make use of the set of adapted coordinates $\widetilde{\mathtt{y}}:=\mathcal{L}_\mathtt{a}(\widetilde{x})$ for the curve $\widetilde{\gamma}^\star$, as defined in paragraph 6.2.3. We remind that $\mathcal{L}_\mathtt{a}:=\mathrm{id}$ in case $m=1$ (see also (6.3.5)).

In particular, for all $m\in\{1,\dots,n-1\}$ we write $\mathtt{U}_{r,\mathtt{a}}^\star:=U_r^\star\circ\mathcal{L}_\mathtt{a}^{-1}$ and $\widetilde{\gamma}_\mathtt{a}^\star:=\mathcal{L}_\mathtt{a}\circ\widetilde{\gamma}^\star$.

By construction, the curve $\widetilde{\gamma}_\mathtt{a}^\star$ is analytic in $[\lambda_{min}-t^\star,\lambda_{max}-t^\star]$ with complex analyticity width $\lambda/\mathtt{K}$, and $|\mathtt{I}'_\lambda|=|\lambda_{max}-\lambda_{min}|=\lambda/(3\mathtt{K})$, as $\mathtt{I}'_\lambda$ was obtained by cutting $\mathtt{I}_\lambda$ into three equal pieces and by taking the central one. Hence, $\widetilde{\gamma}_\mathtt{a}^\star\in\Theta_m^1$ following Definition 6.1.1. By (7.2.34) and Lemma 6.3.1, there exist constants[1] $C_1=C_1(\zeta)$ and

[1] In Lemma 6.3.1, C_1 depends on the open set $\mathtt{D}\in\mathcal{P}(r,m)\backslash\Sigma(r,s,m)$. In our case, by (7.2.34), $\mathtt{D}$ is the open ball of radius $\frac{1}{2}\zeta$ around T, which is at distance at least $\frac{1}{2}\zeta$ from $\Sigma(r,s,m)$; hence, with slight abuse of notation, we can write $C_1=C_1(\zeta)$.

$C_2 = C_2(s, m)$ such that one has the lower estimate

$$\begin{aligned} &\max_{\alpha=1,\dots,s} \left| \frac{d^\alpha}{du^\alpha} \left(\frac{\partial \mathtt{U}^\star_{r,\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_1} \bigg|_{\widetilde{\gamma}^\star_{\mathtt{a}}(u)} \right)_{u=0} \right| > C_1 \\ &\text{in case } s = 1 \text{ or } m = 1, \\ &\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{du^\alpha} \left(\frac{\partial \mathtt{U}^\star_{r,\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell} \bigg|_{\widetilde{\gamma}^\star_{\mathtt{a}}(u)} \right)_{u=0} \right| > \frac{C_1}{1 + C_2 \times \max_{\substack{\ell=2,\dots,m \\ \alpha=2,\dots,s}} |a_{\ell\alpha}|} \\ &\text{in case } 2 \le s \le r-1 \text{ and } m \ge 2\,, \end{aligned} \tag{7.2.35}$$

where the $a_{\ell\alpha}$'s, with $\ell \in \{2, ..., m\}$ and $\alpha \in \{2, ..., s\}$, are the Taylor coefficients of $\widetilde{\gamma}^\star_{\mathtt{a}}(u)$ at the origin. Definition (7.2.22) assures that the Taylor coefficients of order equal or higher than one of the curve $\widetilde{\gamma}^\star(u)$ at $u = 0$, and those of the curve γ calculated at $t = t^\star$ coincide. Moreover, by construction (see paragraph (6.2.3)) $\widetilde{\gamma}^\star_{\mathtt{a}}(u)$ and $\widetilde{\gamma}^\star(u)$ share the same Taylor coefficients of order greater or equal than two calculated at the origin. Hence, the Bernstein estimate in (5.0.1) applied to the second relation in (7.2.35) and the fact that $0 < \lambda \le \lambda_0 \le 1$ (see (7.2.33)) yield that there exists a uniform constant $\mathtt{M} = \mathtt{M}(r', n, m, s) \ge 1$ such that estimate

$$\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{du^\alpha} \left(\frac{\partial \mathtt{U}^\star_{r,\mathtt{a}}(\mathtt{y})}{\partial \mathtt{y}_\ell} \bigg|_{\widetilde{\gamma}^\star_{\mathtt{a}}(u)} \right)_{u=0} \right| > \frac{C_1}{1 + C_2\, \mathtt{M}}\, \lambda^{s-1} \tag{7.2.36}$$

holds in case $2 \le s \le r-1$ and $m \ge 2$.

Now, expression (6.2.29) together with estimate (5.0.1) yields

$$||\mathcal{L}_{\mathtt{a}}^{-1}||_\infty \le 1 + \mathtt{M}\,,$$

for the matrix norm of $\mathcal{L}_{\mathtt{a}}^{-1}$. Therefore, by (6.2.31), by the first estimate in (7.2.35) and by (7.2.36), we infer

$$\begin{aligned} &\max_{\alpha=1,\dots,s} \left| \frac{d^\alpha}{du^\alpha} \left(\frac{\partial U^\star_r(x)}{\partial x_1} \bigg|_{\widetilde{\gamma}^\star(u)} \right)_{u=0} \right| > \frac{C_1}{1 + \mathtt{M}} \\ &\text{for } s = 1 \text{ or } m = 1, \\ &\max_{\substack{\ell=1,\dots,m \\ \alpha=1,\dots,s}} \left| \frac{d^\alpha}{du^\alpha} \left(\frac{\partial U^\star_r(x)}{\partial x_\ell} \bigg|_{\widetilde{\gamma}^\star(u)} \right)_{u=0} \right| > \frac{C_1}{(1 + C_2\mathtt{M})(1 + \mathtt{M})}\lambda^{s-1} \\ &\text{for } 2 \le s \le r-1,\ m \ge 2\,. \end{aligned} \tag{7.2.37}$$

Since in expression (7.2.37) one is considering only the derivatives up to order $s \in \{1 \dots, r-1\}$ at the origin $u = 0$ and $\widetilde{\gamma}^\star(u)$ contains no constant terms, the same estimate holds true for the polynomial $U^\star$.

The thesis follows from (7.2.37) and from (7.2.25) by setting $C' = \dfrac{C_1}{(1+C_2\mathtt{M})(1+\mathtt{M})}$.
□

With the help of Lemmata 7.2.2 and 7.2.3, we are now able to prove Theorem 7.2.1.

Proof. (Theorem 7.2.1)
Introduction. We assume the setting and the notations of Lemmata 7.2.2-7.2.3. In particular, for $0 < \varepsilon \le \varepsilon^\star/2$ we consider a polynomial $S \in \mathcal{P}^\star(r,n)$ in the ball $B^N(Q,\varepsilon)$, and a given m-dimensional subspace Γ^m orthogonal to $\nabla S(0) \neq 0$. We denote by γ the minimal arc of Theorem 5.0.1 - whose image is contained in the thalweg $\mathcal{T}(S,\Gamma^m)$ - and for any $0 < \lambda \le \lambda_0 = \lambda_0(r,s,m,\tau)$ we indicate by $\mathtt{I}_\lambda$ its interval of analyticity of length $\lambda/\mathtt{K}$, where $\mathtt{K} = \mathtt{K}(r',n,m)$ is a uniform constant. We also indicate by $\mathtt{I}'_\lambda$ the interval which is obtained by dividing $\mathtt{I}_\lambda$ into three equal parts and by taking the central one.

Finally, we set $T(x) := S|_{\Gamma^m}(x) := S(Ax)$, with $A \in O(n,m)$ a matrix belonging to the image of the continuous section $\xi_i : \mathcal{E}_i(\Gamma^m) \longrightarrow O(n,m)$, with $i \in \{1,...,L\}$, and whose columns span Γ^m (see the proof of Lemma 7.2.2).

We proceed by steps.

Step 1. For $\ell = 1,...,m$ and $\alpha = \{1,...,s\}$, we consider the functions

$$g_\ell^{(\alpha)}(t^\star) := \frac{d^\alpha}{dt^\alpha}\left(\frac{\partial T(x)}{\partial x_\ell}\bigg|_{\gamma(t)}\right)_{t=t^\star} \quad , \quad t^\star \in \mathtt{I}'_\lambda \tag{7.2.38}$$

and the constant functions

$$g_{m+1}(t^\star) := C' \; , \quad g_{m+2}(t^\star) := -C' \; , \quad g_{m+3}(t^\star) := C'\lambda^{s-1} \; , \quad g_{m+4}(t^\star) := -C'\lambda^{s-1} \; .$$

The degree of T is bounded by r' and - on the interval $\mathtt{I}_\lambda$ - γ is an analytic-algebraic function whose diagram is bounded by a positive integer $\mathtt{d} = \mathtt{d}(r',n,m)$ (see Point 2 of the thesis in Theorem 5.0.1). Hence, for any given choice of $\alpha \in \{1,...,s\}$ and $\ell \in \{1,...,m\}$, the function $g_\ell^{(\alpha)}(t^\star)$ is Nash (i.e. semi-algebraic of class C^∞) due to Propositions A.1.8 and A.1.10, and its diagram is bounded by a quantity depending only on r',m,n. In addition, Proposition A.1.5 ensures that $g_\ell^{(\alpha)}(t^\star)$ is actually analytic-algebraic in $\mathtt{I}'_\lambda$. The same is obviously true also for $g_{m+j}(t^\star)$ for $j \in \{1,2,3,4\}$. Therefore, we set $\widehat{d} = \widehat{d}(r',m,n) := \max_{i\in\{1,...,m+4\}} \max_{\alpha\in\{1,...,s\}}\{\mathrm{diag}\,(g_i^{(\alpha)})\}$.

Now, for any choice of $\alpha \in \{1,...,s\}$ and $i \in \{1,...,m+4\}$, the graph of $g_i^{(\alpha)}$ over $\mathtt{I}'_\lambda$ belongs to the algebraic curve of some non-constant polynomial $V_i^{(\alpha)} \in \mathbb{R}[x,y]$ of two variables, whose degree depends on $\widehat{d}(r',m,n)$. If we indicate by

$$V_i^{(\alpha)}(x,y) = \Pi_{k=1}^{K(i,\alpha)}\left(V_{i,k}^{(\alpha)}(x,y)\right)^{p_k} \tag{7.2.39}$$

the decomposition of $V_i^{(\alpha)}(x,y)$ into its irreducible factors, by Bézout's Theorem (see Th. C.2.2) the irreducible components $\left\{(x,y) \in \mathbb{R}^2 | V_{i,k}^{(\alpha)}(x,y) = 0\right\}$ of the algebraic curve $\left\{(x,y)\in\mathbb{R}^2|V_i^{(\alpha)}(x,y)=0\right\}$ intersect at most at a finite number of points which is bounded by $\left(\deg V_i^{(\alpha)}\right)^2$, which in turn is a quantity depending only on $\widehat{d}(r',m,n)$. This fact, together with the regularity of $g_i^{(\alpha)}$ in $\mathtt{I}'_\lambda$, implies that there exist two positive integers $\overline{k}(i,\alpha)\in\{1,\dots,K(i,\alpha)\}$, $w=w(r',m,n)$, and a subinterval of $\mathtt{I}^\star_{\lambda,i,\alpha}\subset\mathtt{I}'_\lambda$ of length $|\mathtt{I}'_\lambda|/w$ verifying

$$\begin{aligned}&\operatorname{graph}\left(g_i^{(\alpha)}\Big|_{\mathtt{I}^\star_{\lambda,i,\alpha}}\right)\subset\left\{(x,y)\in\mathbb{R}^2|V^{(\alpha)}_{i,\overline{k}(i,\alpha)}(x,y)=0\right\}\\&\operatorname{graph}\left(g_i^{(\alpha)}\Big|_{\mathtt{I}^\star_{\lambda,i,\alpha}}\right)\cap\left\{(x,y)\in\mathbb{R}^2|V^{(\alpha)}_{i,k}(x,y)=0\right\}=\varnothing\end{aligned}\tag{7.2.40}$$

for all $k\in\{1,\dots,K(i,\alpha)\}\backslash\{\overline{k}(i,\alpha)\}$.

The above reasoning can be repeated for all other pairs of integers belonging to $\{1,\dots,s\}\times\{1,\dots,m+4\}$ and which are different from (α,i). Hence, finally there exists an interval $\mathtt{I}^\star_\lambda\subset\mathtt{I}'_\lambda$ of length $|\mathtt{I}'_\lambda|/(w^{(m+4)\,s})$ on which for any $(i',\alpha')\in\{1,\dots,s\}\times\{1,\dots,m+4\}$ the relations

$$\begin{aligned}&\operatorname{graph}\left(g_{i'}^{(\alpha')}\Big|_{\mathtt{I}^\star_{\lambda}}\right)\subset\left\{(x,y)\in\mathbb{R}^2|V^{(\alpha')}_{i',\overline{k}(i',\alpha')}(x,y)=0\right\}\\&\operatorname{graph}\left(g_{i'}^{(\alpha')}\Big|_{\mathtt{I}^\star_{\lambda}}\right)\cap\left\{(x,y)\in\mathbb{R}^2|V^{(\alpha')}_{i',k}(x,y)=0\right\}=\varnothing\end{aligned}\tag{7.2.41}$$

are verified for some integer $\overline{k}(i',\alpha')\in\{1,\dots,K(i',\alpha')\}$ and for any integer $k\in\{1,\dots,K(i',\alpha')\}$, with $\overline{k}(i',\alpha')\neq k$.

Then, by (7.2.41) and again by Bézout's Theorem, there exists a positive integer $\mathsf{N}=\mathsf{N}(r',m,n)$ such that for any distinct pairs of integers (α,i) and (β,j) belonging to $\{1,\dots,s\}\times\{1,\dots,m+4\}$, the algebraic curves $\{(x,y)\in(\mathtt{I}^\star_\lambda,\mathbb{R})|V^{(\alpha)}_{i,\overline{k}(i,\alpha)}(x,y)=0\}$ and $\{(x,y)\in(\mathtt{I}^\star_\lambda,\mathbb{R})|V^{(\alpha)}_{i,\overline{k}(i,\alpha)}(x,y)=0\}$ either coincide or intersect at most at $\mathsf{N}=\mathsf{N}(r',m,n)$ points.

By repeating this reasoning for all possible distinct pairs and by taking (7.2.41) into account, one finally has that there exists a positive constant $\mathsf{M}=\mathsf{M}(r',s,m,n)$ and an interval $\mathtt{J}^\star_\lambda$ of uniform length $|\mathtt{J}^\star_\lambda|=|\mathtt{I}^\star_\lambda|/\mathsf{M}$ over which the graphs of any pair of functions among $g_1^{(1)},\dots,g_m^{(1)},\dots,g_1^{(s)},\dots,g_m^{(s)},g_{m+1},\dots,g_{m+4}$ either do not intersect or coincide.

These reasonings - together with the fact that expression (7.2.18) in Lemma 7.2.3 holds for all $t^\star\in\mathtt{J}^\star_\lambda\subset\mathtt{I}'_\lambda$ - yield that there must exist $\overline{\alpha}\in\{1,\dots,s\}$ and $\overline{\ell}\in\{1,\dots,m\}$ verifying

$$\begin{aligned}&\min_{t^\star\in\mathtt{J}^\star_\lambda}\left|g_{\overline{\ell}}^{(\overline{\alpha})}(t^\star)\right|>C'\qquad\text{for } m=1\\&\min_{t^\star\in\mathtt{J}^\star_\lambda}\left|g_{\overline{\ell}}^{(\overline{\alpha})}(t^\star)\right|>C'\lambda^{s-1}\qquad\text{for }\ m\geq2\,.\end{aligned}\tag{7.2.42}$$

Step 2. We apply Lemma 7.2.1 to $g_{\overline{\ell}}$, with $q \equiv \overline{\alpha}$, $[a,b] \equiv \mathtt{J}^{\star}_{\lambda}$ and with β equal to the r.h.s. of (7.2.42). If we ask for

$$4\left(\overline{\alpha}!\frac{\rho}{2\beta}\right)^{1/\overline{\alpha}} \leq \frac{|\mathtt{J}^{\star}_{\lambda}|}{2} = \frac{\lambda}{6\,\mathtt{K}\,\mathsf{M}\,\varpi^{(m+4)\,s}} \tag{7.2.43}$$

and we take into account the fact that $\overline{\alpha} \in \{1, ..., s\}$, we can choose

$$\rho = \frac{2\,\lambda^s}{s\,! \times \left[24\,\mathtt{K}\,\mathsf{M}\,\varpi^{(m+4)\,s}\right]^s} \times \begin{cases} C' & \text{for } m = 1 \\ C'\,\lambda^{s-1} & \text{for } m \geq 2\,. \end{cases} \tag{7.2.44}$$

Hence, in a closed set $\mathtt{A}_\lambda \subset \mathtt{J}^{\star}_{\lambda}$ of measure $\frac{|\mathtt{J}^{\star}_{\lambda}|}{2} = \frac{\lambda}{6\,\mathtt{K}\,\mathsf{M}\,\varpi^{(m+4)\,s}}$, one has

$$|g_{\overline{\ell}}(t)| > \rho = \begin{cases} \mathtt{C}_1 \lambda^s \quad , & \text{for } m = 1 \\ \mathtt{C}_m \lambda^{2s-1} \quad , & \text{for } m \geq 2 \end{cases} \qquad \forall t \in \mathtt{A}_\lambda \tag{7.2.45}$$

for some $\overline{\ell} \in \{1, ..., m\}$, and for a constant

$$\mathtt{C}_m = \mathtt{C}_m(r', r, s, \tau, n) = \frac{2C'(r', r, s, m, \tau)}{s\,! \times \left[24\,\mathtt{K}(r', m, n)\,\mathsf{M}(r', s, m, n)\,\varpi(r', m, n)^{(m+4)\,s}\right]^s}\,, \tag{7.2.46}$$

where $m \in \{1, \ldots, n-1\}$.

Step 3. Taking the definition of $\mathtt{A}_\lambda$ into account, by construction (see (7.2.38)) we have

$$\max_{t\in\mathtt{A}_\lambda} \left|g_{\overline{\ell}}(t)\right| := \max_{t\in\mathtt{A}_\lambda} \left| \left.\frac{\partial T(x)}{\partial x_{\overline{\ell}}}\right|_{\gamma(t)} \right| := \max_{t\in\mathtt{A}_\lambda} \left| \left.\frac{\partial S|_{\Gamma^m}(x)}{\partial x_{\overline{\ell}}}\right|_{\gamma(t)} \right| . \tag{7.2.47}$$

Due to Theorem 5.0.1, the image of γ is contained in the thalweg $\mathcal{T}(S, \Gamma^m)$, that is in the locus of minima of $T := S|_{\Gamma^m}$ on the spheres $\mathcal{S}^m_\eta \subset \Gamma^m$ of radius $\eta > 0$ centered at the origin. Moreover, the curve γ was constructed by a uniform local inversion theorem applied to the curve ϕ of Lemma 5.0.2 that was parametrized by the radius $\eta > 0$ of the spheres $\mathcal{S}^m_\eta \subset \Gamma^m$ and shared the same image with γ. So, to any value of $t \in \mathtt{A}_\lambda$ there corresponds a unique radius $\eta(t)$ associated to a sphere $\mathcal{S}^m_{\eta(t)} \subset \Gamma^m$.

Hence, taken any pair of real numbers λ, ξ satisfying $0 < \lambda \leq \xi \leq \lambda_0$ - where λ_0 is the quantity defined in Lemma 7.2.3 - by the discussion at step 3 of the proof of Theorem 5.0.1 (in particular, the inclusions in (5.0.22)), one has that the inverse image of $\mathtt{A}_\lambda$ is contained in the interval $\mathcal{I}_\xi \subset [0, \xi]$ defined in Lemma 5.0.2. This argument and (7.2.47) imply that for some $\overline{\ell} \in \{1, ..., m\}$ one has

$$\max_{t\in\mathtt{A}_\lambda} \left| \left.\frac{\partial S|_{\Gamma^m}(x)}{\partial x_{\overline{\ell}}}\right|_{\gamma(t)} \right| \leq \max_{\eta\in[0,\xi]} \left| \left.\frac{\partial S|_{\Gamma^m}(x)}{\partial x_{\overline{\ell}}}\right|_{\phi(\eta)} \right| = \max_{\eta\in[0,\xi]} \min_{||x||_2=\eta} \left| \frac{\partial S|_{\Gamma^m}(x)}{\partial x_{\overline{\ell}}} \right| \tag{7.2.48}$$

which in turn, as $0 < \lambda \le \xi \in (0, \lambda_0]$, by taking (7.2.45)-(7.2.47) and the equivalence of norms into account, implies that

$$\max_{\eta\in[0,\xi]} \min_{||x||_2=\eta} \left|\left|\nabla S|_{\Gamma^m}(x)\right|\right|_2 > \begin{cases} \mathtt{C}_1\lambda^s \,, & \text{for } m=1 \\ \mathtt{C}_m\lambda^{2s-1} \,, & \text{for } m \ge 2 \end{cases} \,, \quad \forall\, 0<\lambda\le\xi\in(0,\lambda_0]\,. \tag{7.2.49}$$

Since the coordinates x are associated to an orthonormal basis spanning Γ^m, for any point $I \in \mathbb{R}^n$ contained in the subspace Γ^m one has $\pi_{\Gamma^m}(\nabla_I S(I)) \equiv \nabla_x(S|_{\Gamma^m})(x)$, and by choosing $\lambda = \xi$ in (7.2.49), we have proved that any polynomial $S \in \mathcal{P}^\star(r', n)$ in the ball $B^N(Q, \varepsilon)$, with $\varepsilon \le \varepsilon^\star/2$, is steep at the origin on the subspaces of dimension m with index bounded as in (7.2.2) and with coefficients $\mathtt{C}_m$, λ_0. It remains to prove that this holds true also in a neighborhood of the origin.

Step 4. For any polynomial $S \in \mathcal{P}^\star(r', n)$, we consider the translation

$$\mathtt{H}^\star : \mathcal{P}^\star(r', n) \times \mathbb{R}^n \longrightarrow \mathcal{P}^\star(r', n)\,, \quad (S(I), I^\star) \longmapsto S(I + I^\star)\,. \tag{7.2.50}$$

$\mathtt{H}^\star$ is uniformly continuous over the compact $\overline{B}^N(Q, \varepsilon^\star/4) \times \overline{B}^n(0, 1)$. Hence, there exists a number $\hat{\delta} = \hat{\delta}(\varepsilon^\star) > 0$ such that for any $S \in \overline{B}^N(Q, \varepsilon^\star/4)$, one has

$$\mathtt{H}^\star(\{S\} \times \overline{B}^n(0, \hat{\delta})) \subset B^N(S, \varepsilon^\star/4)\,. \tag{7.2.51}$$

Hence, for any given point $I^\star$ satisfying $||I^\star||_2 < \hat{\delta}$ and for any polynomial $S \in \overline{B}^N(Q, \varepsilon^\star/4)$, the polynomial $S(I + I^\star)$ belongs to $B^N(Q, \varepsilon^\star/2)$.

Now, we consider a polynomial $S \in \overline{B}^N(Q_r, \varepsilon^\star/4)$. By the above reasonings, for any $I^\star$ verifying $||I^\star||_2 < \hat{\delta}$, one has that its translation $S(I + I^\star)$ belongs to $B^N(Q, \varepsilon^\star/2)$. We have proved at Step 3 that any polynomial in $\mathcal{P}^\star(r', n)$ belonging to $B^N(Q, \varepsilon)$ - with $\varepsilon \in [0, \varepsilon^\star/2]$ - is steep at the origin on the subspaces of dimension m, with index as in (7.2.2), and with uniform coefficients $\mathtt{C}_m$, λ_0. Consequently, for any given $I^\star$ satisfying $||I^\star||_2 < \hat{\delta}$, the polynomial $S(I + I^\star)$ is steep at the origin on the m-dimensional subspaces, with uniform index and coefficients. This is equivalent to stating the same property for polynomial S at any point $I^\star$ in a ball of radius $\hat{\delta}$ around the origin.

The thesis follows by setting $\varepsilon_0 = \varepsilon_0(r, s, m, \tau, n) := \varepsilon^\star/4$.

□

7.3 Proof of the genericity of steepness

With the help of Theorem 7.2.1, we are finally able to prove Theorem A.

Proof. (*Theorem A*)

It is sufficient to study the case in which $I_0 = 0$, else one considers the translated function $h_0(I) := h(I + I_0)$. We proceed by steps.

Step 1. For any choice of integers $r, n \geq 2$, and for any given $\mathtt{s} = (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$, where $s_m \in \{1, \dots, r-1\}$ for all $m \in \{1, \dots, n-1\}$, by taking (7.1.2) into account, we define

$$\mathbf{\Omega}_n^{r,\mathtt{s}} := \bigcup_{m=1}^{n-1} \mathcal{V}(r, s_m, m, n) \subset \mathcal{P}^\star(r, n) \,. \tag{7.3.1}$$

The above set is closed due to Lemma 7.1.1.

For any given pair $\varrho, \tau > 0$, we consider a function $h \in \mathscr{D} \subset C_b^{2r-1}(\overline{B}^n(0, \varrho))$ satisfying

$$\nabla h(0) \neq 0 \quad , \qquad \left|\left|\mathtt{T}_0(h, r, n) - \mathbf{\Omega}_n^{r,\mathtt{s}}\right|\right|_\infty > \tau \,. \tag{7.3.2}$$

Now, for $m = 1, \dots, n-1$, taking the definition of $\varepsilon_0(r, s_m, m, \tau, n)$ in Theorem 7.2.1 into account, we set

$$\overline{\epsilon} = \overline{\epsilon}(r, \mathtt{s}, \tau, n) := \frac{1}{2} \times \min_{m \in \{1, \dots, n-1\}} \{\varepsilon_0(r, s_m, m, \tau, n) > 0\} \,. \tag{7.3.3}$$

Then, for $\varepsilon \in [0, \overline{\epsilon}]$, we consider a function $f \in \mathscr{D} \subset C_b^{2r-1}(\overline{B}^n(0, \varrho))$ satifying

$$f \in \mathfrak{B}^{2r-2}(h, \varepsilon, \overline{B}^n(0, \varrho)) \,. \tag{7.3.4}$$

Due to (7.3.4), $\mathtt{T}_0\,(f, 2r-2, n)$ is contained in a ball of radius ε around $\mathtt{T}_0\,(h, 2r-2, n)$ in $\mathcal{P}(r, n)$. Hence, as by construction we have set $\varepsilon \leq \overline{\epsilon}$, where $\overline{\epsilon}$ was defined in (7.3.3), the definition of set $\mathbf{\Omega}_n^{r,\mathtt{s}}$ in (7.3.1), together with condition (7.3.2) yields that we can apply Theorem 7.2.1 with $r' = 2r - 2$. In turn, this ensures the existence of positive constants $\mathtt{C}_m = \mathtt{C}_m(r' = 2r-2, r, s_m, \tau, n)$,

$$\widehat{d} = \widehat{d}(r, \mathtt{s}, \tau, n) := \min_{m \in \{1, \dots, n-1\}} \widehat{\delta}(r, s_m, m, \tau, n) \,,$$

and

$$\overline{\lambda} = \overline{\lambda}(r, \mathtt{s}, \tau) := \min_{m \in \{1, \dots, n-1\}} \lambda_0(r, s_m, m, \tau)$$

such that $\mathtt{T}_0(f, 2r-2, n)$ is steep in an open ball of radius $\widehat{d}$ around the origin with coefficients $\overline{\lambda}, \mathtt{C}_m$, $m = 1, \dots, n-1$, and with indices

$$\overline{\alpha}_m(s_m) := \begin{cases} s_1 \quad , & \text{if } m = 1 \\ 2s_m - 1 \quad , & \text{if } m \geq 2 \;. \end{cases} \tag{7.3.5}$$

Step 2. For any $I \in B^n(0, R)$ - with

$$R = R(r, \mathtt{s}, \tau, n, \varrho) := \min \left\{ \frac{\varrho}{3}, \frac{\widehat{d}(r, \mathtt{s}, \tau, n)}{2} \right\} \,, \tag{7.3.6}$$

for any $m \in \{1, \dots, n-1\}$, and for any m-dimensional affine subspace $\Gamma^m = \Gamma^m(I)$ passing through I and orthogonal to $\nabla f(I) \neq 0$, we indicate by $f|_{\Gamma^m}$ the restriction of f to Γ^m. We assume that any given $\Gamma^m(I)$ is endowed with the induced euclidean metric, and we indicate by x a suitable system of coordinates on $\Gamma^m(I)$ whose origin $x = 0$ corresponds to point I. Moreover, for any $\beta \in \{1, \dots, n\}$, we set $\partial_\beta := \frac{\partial}{\partial x_\beta}$ and

$$\kappa := \min\left\{\overline{\lambda}, \frac{\varrho}{3}\right\} ,$$

Now, we fix both $I \in B^n(0, R)$ and $\Gamma^m(I)$. By standard calculus, at any point x verifying $||x|| \leq \kappa$ (hence, sufficiently close to I), one can write

$$\begin{aligned}|\partial_\beta(f|_{\Gamma^m})(x) - \mathtt{T}_0(\partial_\beta(f|_{\Gamma^m}), 2r-3, m)(x)|&\\ \leq K(r,m) \max_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|=2r-1}} \max_{I'\in\overline{B}^n(0,\varrho)} |D^\alpha f(I')| \frac{||x||_2^{2r-2}}{(2r-2)!}&\end{aligned} \tag{7.3.7}$$

for some constant $K(r, m) > 0$.

Since $\mathtt{T}_0\,(\partial_\beta(f|_{\Gamma^m}), 2r-3, m)(x) = \partial_\beta[\mathtt{T}_0(f|_{\Gamma^m}, 2r-2, m)](x)$, taking (7.3.7) into account, we have

$$\begin{aligned}&|\partial_\beta(f|_{\Gamma^m})(x)|\\ &\geq \Big|\, |\,\partial_\beta[\,\mathtt{T}_0\,(f|_{\Gamma^m}, 2r-2, m)\,]\,(x)\,| - |\partial_\beta(f|_{\Gamma^m})(x) - \mathtt{T}_0(\partial_\beta(f|_{\Gamma^m}), 2r-3, m)\,(x)|\,\Big|\\ &\geq |\,\partial_\beta[\,\mathtt{T}_0\,(f|_{\Gamma^m}, 2r-2, m)\,]\,(x)\,| - c(r,n,m,\varrho,\mathscr{D})\,||x||_2{}^{2r-2} ,\end{aligned} \tag{7.3.8}$$

where we have indicated

$$c = c(r,n,m,\varrho,\mathscr{D}) := \frac{K(r,m)}{(2r-2)!} \max_{g\in\mathscr{D}} ||g||_{C_b^{2r-1}(\overline{B}^n(0,\varrho))} .$$

Estimate (7.3.8) implies that for any $x \in \Gamma^m(I)$ verifying $||x|| \leq \kappa$ we can write

$$||\nabla(f|_{\Gamma^m})(x)||_1 \geq \left|\left|\nabla\mathtt{T}_0\,(f|_{\Gamma^m}, 2r-2, m)(x)\right|\right|_1 - c\,n\,||x||_2{}^{2r-2} ,$$

and, by the equivalence of norms,

$$||\nabla(f|_{\Gamma^m})(x)||_2 \geq \frac{1}{n}\left|\left|\nabla\mathtt{T}_0\,(f|_{\Gamma^m}, 2r-2, m)(x)\right|\right|_2 - c\,||x||_2{}^{2r-2} . \tag{7.3.9}$$

Step 3. By the discussion at Step 1, $\mathtt{T}_0(f, 2r-2, n)$ is steep in an open ball of radius $\widehat{d}$ around the origin $I = 0$, with coefficients $\bar{\lambda}, \mathtt{C}_m$, $m = 1, \dots, n-1$, and with indices as in (7.3.5). This property, together with expression (7.3.9) and with the fact that

- the origin $x = 0$ on $\Gamma^m(I)$ corresponds to point $I \in B^n(0, R)$ by construction;
- $R \leq \widehat{d}/2$ by (7.3.6);

yields

$$\max_{\eta\in[0,\xi]} \min_{\substack{||x||_2\in\Gamma^1(I)\\ ||x||_2=\eta}} ||\nabla(f|_{\Gamma^1})(x)||_2 > \frac{\mathtt{C}_1}{n}\xi^{s_1} - c\,\xi^{2r-2} \quad \forall\xi\in[0,\kappa] \quad (m=1) \quad (7.3.10)$$

$$\max_{\eta\in[0,\xi]} \min_{\substack{||x||_2\in\Gamma^m(I)\\ ||x||_2=\eta}} ||\nabla(f|_{\Gamma^m})(x)||_2 > \frac{\mathtt{C}_m}{n}\xi^{2s_m-1} - c\,\xi^{2r-2} \quad \forall\xi\in[0,\kappa] \quad (2\le m\le n-1)\,. \quad (7.3.11)$$

If we impose

$$\begin{cases} c\,\xi^{2r-2} \le \dfrac{\mathtt{C}_1}{2n}\xi^{s_1} \quad , & \text{if } m=1 \\ c\,\xi^{2r-2} \le \dfrac{\mathtt{C}_m}{2n}\xi^{2s_m-1} \quad , & \text{if } m=2,\dots,n-1 \end{cases} \quad (7.3.12)$$

by (7.3.10)-(7.3.11), and by the fact that $s_m \le r-1$ for all $m=1,\dots,n-1$, we have that f is steep in a ball of radius R around the origin with coefficients (we have set $r' = 2r-2$)

$$\begin{aligned} \delta &= \delta(r,\mathtt{s},\tau,n,\varrho,\mathscr{D}) \\ &:= \min\left\{\kappa, \left(\frac{\mathtt{C}_1(r',r,s_1,\tau,n)}{2\,n\,c(r,n,m,\varrho,\mathscr{D})}\right)^{\frac{1}{2r-2-s_1}}, \min_{m\in\{2,\dots,n-1\}}\left\{\left(\frac{\mathtt{C}_m(r',r,s_m,\tau,n)}{2\,n\,c(r,n,m,\varrho,\mathscr{D})}\right)^{\frac{1}{2(r-s_m)-1}}\right\}\right\}, \end{aligned} \quad (7.3.13)$$

$$C_m(r',r,s_m,\tau,n) := \frac{\mathtt{C}_m(r',r,s_m,m,\tau,n)}{2n}\,,$$

and with indices bounded as in (7.3.5).

It remains to prove the estimate on the codimension of $\mathbf{\Omega}_n^{r,\mathtt{s}}$. By (7.3.1), Lemma 7.1.2 and Proposition A.1.3 we have

$$\operatorname{codim} \mathbf{\Omega}_n^{r,\mathtt{s}} \ge \max\left\{0, \min_{m\in\{1,\dots,n-1\}}\{s_m - m(n-m-1)\}\right\}.$$

This concludes the proof.

$\square$

Chapter 8

Proof of Theorem B and of its Corollaries

Hereafter, we assume the notations and the results of the previous sections.

8.1 Proof of Theorem B

It suffices to prove the statement for $I_0 = 0$, otherwise one considers $h_0(I) := h(I+I_0)$.

8.1.1 Case $m = 1$.

Let Γ^1 be a 1-dimensional subspace (a line) orthogonal to $\nabla h(0) \neq 0$, and let $w \in \mathbb{S}^n$ be its generating vector. By standard results of calculus, the restriction of the Taylor polynomial $\mathtt{T}_0(h, r, n)$ to Γ^1, indicated by $\mathtt{T}_0(h|_{\Gamma^m}, r, 1)$, reads

$$\mathtt{T}_0(h|_{\Gamma^1}, r, 1)(x) = h(0) + \sum_{i=1}^{r} \frac{1}{i!} h^i[w, \dots, w] x^i \ , \tag{8.1.1}$$

where the notation in (3.0.2) has been taken into account, and where x is the coordinate associated to the vector w.

By (8.1.1) and Lemma 6.2.1, condition (4.2.8) amounts to asking that, for any subspace Γ^1, the polynomial $\mathtt{T}_0(h|_{\Gamma^1}, r, 1)$ belongs to the complementary of the set of s_1-vanishing polynomials $\sigma(r, s_1, 1)$ in $\mathcal{P}(r, 1)$. Moreover, again by Lemma 6.2.1, one has $\sigma(r, s_1, 1) = \overline{\sigma}(r, s_1, 1) =: \Sigma(r, s_1, 1)$. Hence, by definitions (7.1.1)-(7.1.2), by Theorem A and by (7.3.1), h is steep on the subspaces of dimension one in a neighborhood of the origin, with steepness index bounded by s_1.

8.1.2 Case $n \geq 3, 2 \leq m \leq n-1$.

It is sufficient to prove that, for fixed $m \in \{1, \dots, n-1\}$, under the assumptions at point $ii)$ of Theorem B, one has $\mathtt{T}_0(h, r, n) \in \mathcal{P}(r, n) \backslash \mathcal{V}(r, s_m, m, n)$, where the set $\mathcal{V}(r, s_m, m, n)$ was defined in (7.1.2). The thesis then follows by Theorem A and by expression (7.3.1).

By absurd, suppose that the claim is false. Then, by (7.1.1)-(7.1.2), there exists some subspace Γ^m orthogonal to $\nabla h(0) \neq 0$ such that $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \Sigma(r, s_m, m)$.

Hence, since $\Sigma(r, s_m, m) := \overline{\sigma}(r, s_m, m)$ by construction, there are two possibilities:

1. either $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \sigma(r, s_m, m)$;

2. or $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \Sigma(r, s_m, m) \backslash \sigma(r, s_m, m)$.

We consider the two cases separately and we prove that in both cases we end up being in contradiction with the hypotheses.

Case 1. If $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \sigma(r, s_m, m)$, then by construction $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$ satisfies the s_m vanishing condition at the origin on some curve $\gamma \in \Theta_m$, whose image is contained in Γ^m. Since one is free to choose the orthonormal basis $\{u_1, \dots, u_m\} \in \mathbb{U}(m, n)$ spanning Γ^m, up to a changement in the order of the vectors we can suppose without loss of generality that the coordinate which parametrizes the curve γ is the first one, that is $\gamma \in \Theta^1_m$, and $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \sigma^1(r, s_m, m)$. Moreover, following section 6.2.3, we can make use of the adapted coordinates for the curve γ, which are associated to the basis (see expression (6.2.13))

$$\left\{ v := u_1 + \sum_{i=2}^{m} a_{i1} u_i,\ u_2, \dots, u_m \right\} \in \mathscr{V}^1(m, n) , \tag{8.1.2}$$

where, as we did in 6.2.3, we have indicated by $\mathtt{a} := (a_{21}, \dots, a_{1m}) \in \mathbb{R}^{m-1}$ the vector containing the linear terms of the Taylor expansion of γ at the origin. Following the notations of section 6.2.3 (especially, formula (6.2.16)), we write

$$\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m)(\mathtt{y}) := \mathtt{T}_0(h|_{\Gamma^m}, r, m) \circ \mathcal{L}_{\mathtt{a}}^{-1}(\mathtt{y}) .$$

Then, by standard results of calculus, taking (3.0.2) into account, one can write

$$\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m)(\mathtt{y}) = \sum_{\substack{\mu \in \mathbb{N}^m \\ 1 \leq |\mu| \leq r}} \frac{1}{\mu!} h_0^{|\mu|} \Big[\overbrace{v}^{\mu_1}, \overbrace{u_2}^{\mu_2}, \dots, \overbrace{u_m}^{\mu_m} \Big] \mathtt{y}_1^{\mu_1} \dots \mathtt{y}_m^{\mu_m} , \tag{8.1.3}$$

where $\mu! := \mu_1! \dots, \mu_m!$.

Since $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \sigma^1(r, s_m, m)$, by (6.2.10) and Lemma 6.2.3, one has

$$\mathtt{Q}_{i\alpha}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) = 0 \qquad \forall\, i \in \{1, \dots, m\} ,\ \ \forall \alpha \in \{0, \dots, s\} , \tag{8.1.4}$$

where $\mathtt{J}_{s_m,\gamma,\mathtt{a}} := \mathcal{L}_{\mathtt{a}} \circ \mathcal{J}_{s_m,\gamma}$ denotes the s_m-truncation at the origin of the curve γ expressed in the adapted variables.

We now try to simplify the expressions in (8.1.4), by taking expressions (6.2.39)-(6.2.40) in Lemma 6.2.3 into account and by exploiting the form of the polynomial $\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m)$ in (8.1.3). Namely - thanks to (8.1.3) and (4.2.2) - the coefficients $\mathtt{p}_{\nu(1,\alpha)}$ and $\mathtt{p}_{\nu(\ell,\alpha)}$, with $\ell \in \{2, \dots, m\}$, $\alpha \in \{0, \dots, s_m\}$, appearing in (6.2.39)-(6.2.40) read

$$\mathtt{p}_{\nu(1,\alpha)} = \frac{1}{(\alpha+1)!} h_0^{\alpha+1}[v, \dots, v] \quad , \qquad \mathtt{p}_{\nu(\ell,\alpha)} = \frac{1}{\alpha!} h_0^{\alpha+1}\Big[\overbrace{v}^{\alpha}, u_\ell \Big] \, . \tag{8.1.5}$$

Moreover, if $s_m \geq 2$, for $\alpha \in \{2, \dots, s_m\}$, exploiting (8.1.3) and the linearity, the second addend at the right hand side of (6.2.39) in our case reads

$$\begin{aligned} \sum_{i=2}^{m} \sum_{\beta=1}^{\alpha-1} \beta \, \mathtt{p}_{\nu(i,\beta)} a_{i(\alpha-(\beta-1))} &= \sum_{i=2}^{m} \sum_{\beta=1}^{\alpha-1} \frac{\beta}{\beta!} h_0^{\beta+1}\Big[\overbrace{v}^{\beta}, u_i \Big] a_{i(\alpha-(\beta-1))} \\ &= \sum_{\beta=1}^{\alpha-1} \frac{1}{(\beta-1)!} h_0^{\beta+1}\Big[\overbrace{v}^{\beta}, \sum_{i=2}^{m} a_{i(\alpha-(\beta-1))} u_i \Big] \, . \end{aligned} \tag{8.1.6}$$

Henceforth, in order to simplify our formulas, we make use of the notation

$$\mathbf{v}_i := \begin{cases} v \, , \ \text{if } i = 1 \\ u_i \, , \ \text{if } i \in \{2, \dots, m\} \end{cases} \quad .$$

Considering again the case $s_m \geq 2$, for $\alpha \in \{2, \dots, s_m\}$, and for any $i = 1, \dots, m$, by (8.1.3) and by (4.2.4)-(4.2.5), the last addend at the right hand side of (6.2.39)-(6.2.40) reads

$$\sum_{\substack{\mu\in\mathcal{E}_m(i,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_i\neq 0}} \mu_i\, \mathtt{p}_\mu \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(i),\alpha)} \left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(i)}{k_{j2}\,\dots\,k_{j\alpha}}\, {a_{j2}}^{k_{j2}}\dots{a_{j\alpha}}^{k_{j\alpha}}\right)$$

$$= \sum_{\substack{\mu\in\mathcal{E}_m(i,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_i\neq 0}} \frac{\mu_i}{\mu!} h_0^{|\mu|}\Big[\overbrace{v}^{\widetilde{\mu}_1(i)}, \overbrace{u_2}^{\widetilde{\mu}_2(i)}, \dots, \overbrace{u_m}^{\widetilde{\mu}_m(i)}, \mathbf{V}_i\Big]$$

$$\times \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(i),\alpha)} \left(\prod_{j=2}^{m}\binom{\widetilde{\mu}_j(i)}{k_{j2}\dots k_{j\alpha}}{a_{j2}}^{k_{j2}}\dots{a_{j\alpha}}^{k_{j\alpha}}\right)$$

$$= \sum_{\substack{\mu\in\mathcal{E}_m(i,\alpha)\\ \mu\in\mathcal{M}_m(\alpha)\\ \mu_i\neq 0}} \sum_{k\in\mathcal{G}_m(\widetilde{\mu}(i),\alpha)} \frac{h_0^{|\mu|}\Big[\overbrace{v}^{\widetilde{\mu}_1(i)}, \overbrace{a_{22}\,u_2}^{k_{22}}, \dots, \overbrace{a_{2\alpha}\,u_2}^{k_{2\alpha}}, \overbrace{a_{m2}\,u_m}^{k_{m2}}, \dots, \overbrace{a_{m\alpha}\,u_m}^{k_{m\alpha}}, \mathbf{V}_i\Big]}{\widetilde{\mu}_1(i)!\; k!}\,, \tag{8.1.7}$$

where the last passage is a consequence of the multi-linearity and of the fact that, for all $i\in\{1,\dots,m\}$, $j\in\{2,\dots,m\}$, we have $\sum_{u=2}^{\alpha} k_{ju} = \widetilde{\mu}_j(i)$ by construction (see (4.2.4)).

By (6.2.39)-(6.2.40), it is trivial to observe that for $\alpha = 0$ and for all $i\in\{1,\dots,m\}$, one has

$$\mathtt{Q}_{i,0}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) = 0 \quad\Longleftrightarrow\quad h_0^1[v] = h_0^1[u_2] = \dots h_0^1[u_m] = 0 \tag{8.1.8}$$

which simply means that the basis vectors $\{v, u_2, \dots, u_\ell\}$ are orthogonal to $\nabla h(0)\neq 0$.

We observe that, in case $i = 1,\dots,m$ and $\alpha = 1$, taking (8.1.5) and (6.2.39)-(6.2.40) into account, condition (8.1.4) amounts to requiring that for all $i\in\{1,\dots,m\}$

$$\mathtt{Q}_{i,1}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) = 0 \iff h_0^2[v,v] = h_0^2[v,u_2] = \dots = h_0^2[v,u_m] = 0\,. \tag{8.1.9}$$

For $i = 1$ and $\alpha = 2$, instead, we have

$$\mathtt{p}_{\nu(1,\alpha)} = \frac{1}{3!} h^3[v,v,v] \tag{8.1.10}$$

and for $\beta = 1$ the term in (8.1.6) is null thanks to (8.1.9).

Moreover, still for $i = 1$ and $\alpha = 2$ we observe that also the term in (8.1.7) does not yield any contribution to condition (8.1.4). In order to see this, we start by remarking that the multi-indices to be taken into account in (8.1.7) for $i = 2$ and $\alpha = 2$ must satisfy $\mu\in\mathcal{E}_m(1,2)$, that is, by (4.2.4)-(4.2.5)

$$k_{j2} = \widetilde{\mu}_j(1) \quad \forall\, j\in\{2,\dots,m\} \quad , \qquad \widetilde{\mu}_1(1) + \sum_{j=2}^{m} 2k_{j2} = 2\,. \tag{8.1.11}$$

Conditions (8.1.11) are only possible if

1. $\widetilde{\mu}_1(1) = 2$ and $\widetilde{\mu}_j(1) = 0$ for all $j \in \{2, \dots, m\}$, which implies $\mu = (3, 0, \dots, 0)$.
2. $k_{j2} = \widetilde{\mu}_j(1) = \delta_{jp}$ for some given $p \in \{2, \dots, m\}$ and $\widetilde{\mu}_1(1) = 0$. This implies that $\mu_1 = 1$ and $\mu_j = \delta_{jp}$ for all $j \in \{2, \dots, m\}$, so that finally $|\mu| = 2$.

The first case is incompatible with the condition $\mu \in \mathcal{M}_m(2)$ required in (8.1.7) (see (4.2.3)), as $(3, 0, \dots, 0) \equiv \nu(1, 2)$. The second case does not yield any contribution to (8.1.7) because of (8.1.9).

Hence one has

$$\mathtt{Q}_{1,2}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) = 0 \quad \Longleftrightarrow \quad h^3[v, v, v] = 0\,. \tag{8.1.12}$$

Finally, for $i = 1$ and $\alpha \geq 3$, and for $i \in \{2, \dots, m\}$ and $\alpha \geq 2$, comparing expressions (8.1.5)-(8.1.6)-(8.1.7) with the quantities (6.2.39)-(6.2.40) in Lemma 6.2.3, and taking the definition of the quantities $\mathcal{H}^{h,0}_{m,\ell,\alpha}(v, u_2, \dots, u_m, a(m))$ in (4.2.6)-(4.2.7) into account, one has that

$$\mathtt{Q}_{i,\alpha}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) = 0 \quad \Longleftrightarrow \quad \mathcal{H}^{h,0}_{m,\ell,\alpha}(v, u_2, \dots, u_m, a(m, s_m)) = 0\,. \tag{8.1.13}$$

Putting together (8.1.8)-(8.1.9)-(8.1.12)-(8.1.13) with (8.1.2), we see that the polynomial $\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m)$ belongs to $\sigma^1(r, s_m, m)$ if and only if the system

$$\begin{cases} (u_1, \dots, u_m) \in \mathtt{U}(m, n) \\ a(m) := (a_{21}, \dots, a_{2s_m}, \dots, a_{m1}, \dots, a_{ms_m}) \in \mathbb{R}^{(m-1)\times s_m} \\ v = u_1 + \sum_{j=2}^m a_{j1}\, u_j \\ h^1_0[v] = h^1_0[u_2] = \cdots = h^1_0[u_m] = 0 \\ \mathcal{H}^{h,0}_{m,\ell,\alpha}(v, u_2, \dots, u_m, a(m, s_m)) = 0 \quad i = 1, \dots, m\,, \quad \alpha = 1, \dots, s \end{cases} \tag{8.1.14}$$

is satisfied. However, this is in contradiction with hypothesis (4.2.9) in the statement, therefore $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \notin \sigma^1(r, s_m, m)$.

Case 2. We now assume that $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \Sigma(r, s_m, m)\backslash\sigma(r, s_m, m)$. Up to changing the order of the vectors spanning Γ^m, by (6.3.1) without any loss of generality we can suppose $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \Sigma^1(r, s_m, m)\backslash\sigma(r, s_m, m)$. Then, there must exist a sequence of polynomials $\{P_k \in \sigma^1(r, s_m, m)\}_{k\in\mathbb{N}}$ approaching $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$. To conclude the proof of Case 2, we need the following

Lemma 8.1.1. *There exist a sequence $\{S_k \in \mathcal{P}(r, n)\}_{k\in\mathbb{N}}$ converging to $\mathtt{T}_0(h, r, n)$ in $\mathcal{P}(r, n)$ and verifying $S_k|_{\Gamma^m} = P_k$ for any given $k \in \mathbb{N}$.*

Proof. We indicate by $A_1, \dots, A_m \in \mathtt{U}(m, n)$ an orthonormal basis of Γ^m, and we choose $n - m$ orthonormal supplementary vectors $A_{m+1}, \dots, A_n$ to form a orthonormal basis of $\mathbb{R}^n$. In the coordinates $(x_1, \ldots, x_n)$ associated to $A_1, \ldots, A_n$, the restriction of any polynomial $Q \in \mathcal{P}(r,n)$ to the subspace Γ^m is obtained by simply setting $x_{m+1} = \cdots = x_n = 0$ in the expression of Q. Conversely, any polynomial $P(x_1, \ldots, x_m) \in \mathcal{P}(r,m)$ depending only on the first m variables is the projection on Γ^m of any polynomial $Q \in \mathcal{P}(r,n)$ of the form $Q(x_1, \ldots, x_n) = P(x_1, \ldots, x_m) + q(x_1, \ldots, x_n)$, where $q \in \mathcal{P}(r,n)$ verifies $q(x_1, \ldots, x_m, 0) = 0$.

By the above discussion and with slight abuse of notation, one can define the polynomial

$$q_h = q_h(x_1, \ldots, x_n) := \mathtt{T}_0(h,r,n)(x_1, \ldots, x_n) - \mathtt{T}_0(h|_{\Gamma^m}, r, m)(x_1, \ldots, x_m) \quad (8.1.15)$$

which verifies $q_h(x_1, \ldots, x_m, 0) = 0$ by construction.

Then, for $k \in \mathbb{N}$, we consider the polynomials

$$S_k = S_k(x_1, \ldots, x_n) := P_k(x_1, \ldots, x_m) + q_h(x_1, \ldots, x_n)\,, \quad (8.1.16)$$

where $\{P_k\}_{k\in\mathbb{N}}$ is the sequence approaching $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$ introduced above. The sequence $\{S_k\}_{k\in\mathbb{N}}$ has the properties we seek. Infact, as $q_h(x_1, \ldots, x_m, 0) = 0$, on the one hand S_k verifies

$$S_k|_{\Gamma^m} = S_k(x_1, \ldots, x_m, 0) = P_k \qquad \forall\, k \in \mathbb{N}\,; \quad (8.1.17)$$

on the other hand, as $P_k \longrightarrow \mathtt{T}_0(h|_{\Gamma^m}, r, m)$ by hypothesis, by taking (8.1.15) into account one has

$$S_k \longrightarrow \mathtt{T}_0(h|_{\Gamma^m}, r, m) + q_h = \mathtt{T}_0(h,r,n)\,. \quad (8.1.18)$$

□

Since by Lemma 8.1.1 one has $S_k|_{\Gamma^m} = P_k$, and since $P_k \in \sigma^1(r, s_m, m)$ by construction, the same arguments developed at Case 1 yield that for any $k \in \mathbb{N}$ the system

$$\begin{cases} (u_1, \ldots, u_m) \in \mathtt{U}(m,n) \\ a(m) := (a_{21}, \ldots, a_{2s_m}, \ldots, a_{m1}, \ldots, a_{ms_m}) \in \mathbb{R}^{(m-1)\times s_m} \\ v = u_1 + \sum_{j=2}^m a_{j1}\, u_j \\ (S_k)_0^1[v] = (S_k)_0^1[u_2] = \cdots = (S_k)_0^1[u_m] = 0 \\ \mathcal{H}^{S_k,0}_{m,\ell,\alpha}(v, u_2, \ldots, u_m, a(m,s_m)) = 0 \quad i = 1, \ldots, m\,, \quad \alpha = 1, \ldots, s \end{cases} \quad (8.1.19)$$

must be verified. However, this fact and the fact that, by Lemma 8.1.1, one also has $S_k \longrightarrow \mathtt{T}_0(h,r,n)$, contradicts the hypotheses of Theorem B. Hence, we must have $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \notin \Sigma(r, s_m, m)\backslash\sigma(r, s_m, m)$.

By the discussion at Cases 1-2 above, the assumptions of Theorem B imply that - for any m-dimensional subspace Γ^m, with $m \in \{2, \ldots, n-1\}$ - the Taylor polynomial $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$ lies outside of $\Sigma(r, s_m, m)$. Hence $\mathtt{T}_0(h,r,n) \in \mathcal{P}(r,n)\backslash\mathcal{V}(r, s_m, m, n)$. This, together with (7.3.1) and Theorem A, concludes the proof.

8.2 Proof of the Corollaries

8.2.1 Proof of Corollary B1

We start by studying the one-dimensional affine subspaces orthogonal to $\nabla h(I_0) \neq 0$. Hypothesis (4.2.11) is equivalent to hypothesis (4.2.8) in Theorem B with $r = 3$, $s_1 = 2$, whence the thesis.

On the other hand, for any fixed $m \in \{2, \dots, n-1\}$, since $\mathtt{G}(m, n)$ and $\mathbb{S}^n$ are both compact, by hypothesis (4.2.11) there exists $\tau_m > 0$ such that for any m-dimensional affine subspace $I_0 + \Gamma^m$ orthogonal to $\nabla h(I_0)$, and for any vector $w \in \mathbb{S}^n \cap \Gamma^m$ we have

$$h^1_{I_0}[w] = 0 \quad , \qquad |h^2_{I_0}[w, w]| + |h^3_{I_0}[w, w, w]| \geq \tau_m > 0 \, . \tag{8.2.1}$$

By (8.2.1), h matches the hypotheses at point ii) of Theorem B for $r = 3$ and $s_m = 2$, whence the thesis for affine subspaces of dimension higher or equal than two.

8.2.2 Proof of Corollary B2

In the proof of Theorem A we have set $\mathbf{\Omega}^{r,\mathrm{s}}_n := \bigcup_{m=1}^{n-1} \mathcal{V}(r, s_m, m, n)$ (see (7.3.1)). Furthermore, formula (7.1.2) ensures that for any given function h of class C^{2r-1} around I_0

$$\mathtt{T}_{I_0}(h, r, n) \in \mathcal{V}(r, s_m, m, n) \quad \overset{\text{by definition}}{\Longleftrightarrow} \quad \substack{\exists \Gamma^m \in \mathtt{G}(m,n),\ \Gamma^m \perp \nabla h(I_0) \text{ s.t.} \\ \mathtt{T}_{I_0}(h|_{\Gamma^m}, r, m) \in \Sigma(r, s_m, m) := \overline{\sigma}(r, s_m, m)} \, . \tag{8.2.2}$$

Since one is free to choose the order of the orthonormal vectors spanning Γ^m, without any loss of generality we can also write

$$\mathtt{T}_{I_0}(h, r, n) \in \mathcal{V}(r, s_m, m, n) \quad \Longleftrightarrow \quad \substack{\exists \Gamma^m \in \mathtt{G}(m,n),\ \Gamma^m \perp \nabla h(I_0) \text{ s.t.} \\ \mathtt{T}_{I_0}(h|_{\Gamma^m}, r, m) \in \Sigma^1(r, s_m, m) := \overline{\sigma}^1(r, s_m, m)} \, . \tag{8.2.3}$$

In the proof of Theorem B, we have also seen that for any $m \in \{1, \dots, n-1\}$, and for any given subspace $\Gamma^m \in \mathtt{G}(m, n)$ orthogonal to $\nabla h(I_0)$, condition $\mathtt{T}_{I_0}(h|_{\Gamma^m}, r, m) \in \sigma^1(r, s_m, m)$ holds if and only if system (4.2.12) (if $m = 1$) or (4.2.13) (if $m \geq 2$) admits a solution when P is set to be the Taylor polynomial at order r of function h.

By the above discussion, we have that, for any fixed $m \in \{1, \dots, n-1\}$, condition $\mathtt{T}_{I_0}(h, r, n) \in \mathcal{V}(r, s_m, m, n)$ is equivalent to asking that $\mathtt{T}_{I_0}(h, r, n)$ belongs to the closure in $\mathcal{P}^\star(r, n)$ of the set of polynomials solving system (4.2.12) (if $m = 1$) or system (4.2.13) (for $m \in \{2, \dots, n-1\}$).

The thesis follows by the arguments above and by (7.3.1).

8.2.3 Proof of Corollary B3

By the proof of Theorem B, the conditions in i) and ii) amount to asking for the existence of a real-analytic curve

$$\gamma(t) := \begin{cases} wt & \text{for } m = 1 \\ \left(t, \sum_{i=1}^{+\infty} a_{2i} t^i, \dots, \sum_{i=1}^{+\infty} a_{mi} t^i\right) & \text{for } m = \{2, \dots, n-1\} \end{cases} \tag{8.2.4}$$

whose image is contained in some m-dimensional subspace orthogonal to $\nabla h(I_0) \neq 0$, and such that $(\pi_{\Gamma^m} \nabla h) \circ \gamma$ has a zero of infinite order at $\gamma(0) = I_0$. By analyticity, then, $(\pi_{\Gamma^m} \nabla h) \circ \gamma$ is identically zero. This implies by Definition 11.2.1 that h is not steep.

Chapter 9

Partition of the set of s-vanishing polynomials

The proof of Theorems C1-C2-C3 is quite long and requires intermediate results which will be presented in this section. Before stating them, in the following two paragraphs we will introduce some definitions and notations.

9.1 Initial setting

Consider three integers $r, m \geq 2$, and $s \in \{2, \dots, r-1\}$. In sections 6-7, we have indicated by $\Sigma(r, s, m) \subset \mathcal{P}(r, m)$ the closure of the set $\sigma(r, s, m)$ of s-vanishing polynomials. In particular, by (6.2.9)-(6.2.10) one has

$$\begin{aligned} \sigma(r, s, m) &= \bigcup_{i=1}^{m} \sigma^i(r, s, m) \quad , \qquad \sigma^i(r, s, m) := \Pi_{\mathcal{P}(r,m)} Z^i(r, s, m) \, , \\ \Sigma(r, s, m) &= \bigcup_{i=1}^{n} \Sigma^i(r, s, m) := \bigcup_{i=1}^{n} \overline{\sigma}^i(r, s, m) := \bigcup_{i=1}^{n} \overline{\Pi_{\mathcal{P}(r,m)} Z^i(r, s, m)} \, , \end{aligned} \tag{9.1.1}$$

where the sets $Z^i(r, s, m) \subset \mathcal{P}(r, m) \times \vartheta^i_m(s)$, with $i \in \{1, \dots, m\}$, are defined in (6.2.11), and one has decomposition (6.2.12), namely

$$\mathcal{P}(r, m) \times \vartheta_m(s) \supset Z(r, s, m) =: \bigcup_{i=1}^{m} Z^i(r, s, m) \, .$$

The expression of the sets $Z^i(r, s, m)$, $i \in \{1, \dots, m\}$, is given explicitly [1] in Lemma 6.2.3.

[1] Actually, in Lemma 6.2.3, only the expression of $Z^1(r, s, m)$ is explicit. However, as it was already pointed out in section 6, the cases $i = 2, \dots, m$ are trivial generalizations of the case $i = 1$: in order to find the expression of $Z^i(r, s, m)$ for $i \neq 1$, one simply has to follow the same steps needed to find the expression for $Z^1(r, s, m)$, and to exchange the rôle of the first coordinate with the i-th one.

In the previous sections, we have seen that, in order to check if a given polynomial $Q \in \mathcal{P}(r,n)$ is steep at the origin[2] on a fixed subspace $\Gamma^m \in \mathtt{G}_Q(m,n)$ [3], one must check whether the restriction $P := Q|_{\Gamma^m} \in \mathcal{P}(r,m)$ belongs to the complementary of $\Sigma(r,s,m) := \overline{\sigma}(r,s,m)$. We now claim that it is not strictly necessary to consider the closure of the whole set $\sigma(r,s,m)$. Indeed, in practice, the curves on which the s-vanishing condition must be tested are minimal arcs with uniform characteristics, like the one defined in Theorem 5.0.1.

Namely, by the arguments in the proof of Lemma 7.2.3 - for any given $Q \in \mathcal{P}(r,n)$ and for any fixed $\Gamma^m \in \mathtt{G}_Q(m,n)$, it is sufficient to check if there exists a threshold $\lambda_0 > 0$ such that, for any $0 < \lambda \leq \lambda_0$, there exists an interval $I_\lambda \subset [-\lambda,\lambda]$ of length $\lambda/\mathtt{K}$ [4], on which - for any curve $\gamma \in \Theta_m$ verifying the Bernstein's inequality (5.0.1) - the composition $(\nabla P)\circ\gamma := (\nabla Q|_{\Gamma^m})\circ\gamma$ has no zeros of order greater or equal than s. In particular, we are interested in testing the s-vanishing conditions on those analytic curves γ over I_λ that, for some $i \in \{1,\dots,m\}$, satisfy

$$\gamma(t) = \begin{pmatrix} x_1(t) = \sum_{k=1}^{+\infty} a_{1k}\, t^k \\ \dots \\ x_{i-1}(t) = \sum_{k=1}^{+\infty} a_{(i-1)k}\, t^k \\ x_i(t) = t \\ x_{i+1}(t) = \sum_{k=1}^{+\infty} a_{(i+1)k}\, t^k \\ \dots \\ x_m(t) = \sum_{k=1}^{+\infty} a_{mk}\, t^k \end{pmatrix}, \qquad \max_{u\in I_\lambda} \max_{\substack{j\in\{1,\dots,m\}\\ j\neq i}} |a_{jk}(u)| \leq \frac{\mathtt{M}(r,n,m,k)}{\lambda^{k-1}}\,. \tag{9.1.2}$$

By Theorem 5.0.1, the constant $\mathtt{M} = \mathtt{M}(r,n,m,k)$ in (9.1.2) can be explicitly computed and it is uniform for all curves $\gamma \in \Theta_m$.

As we have shown in the proof of Lemma 7.2.3, for any given $P \in \mathcal{P}(r,m)$, the threshold λ_0 - if it exists - goes to zero with the distance of P to the "bad" set $\sigma(r,s,m)$. Therefore, by formula (9.1.2), the Taylor coefficients of the curves γ on which the s-vanishing condition must be tested may take any value, except for those of order one which, independently from the choice of $\lambda > 0$, are always uniformly bounded by $\mathtt{M}(r,n,m,1)$.

Inspired by the above reasonings, with the notations in (9.1.2), we give the following

Definition 9.1.1. For $i = 1,\dots,m$, we introduce the sets

$$\widehat{\Theta}^i_m := \left\{ \gamma \in \Theta^i_m \,\middle|\, \max_{\substack{j\in\{1,\dots,m\}\\ j\neq i}} \{|a_{j1}(0)|\} \leq \mathtt{M}(r,n,m,1) \right\} \tag{9.1.3}$$

---

[2] It is clear that the arguments devoped in the sequel hold also if the considered point is not the origin.

[3] The symbol $\mathtt{G}_Q(m,n)$ was introduced in the proof of Lemma 7.2.2.

[4] More details about the threshold λ_0 are given in Lemma 7.2.3, whereas the constant $\mathtt{K}$ is the one introduced in Theorem 5.0.1.

and

$$\widehat{\Theta}_m := \bigcup_{i=1}^{m} \widehat{\Theta}^i_m \,, \tag{9.1.4}$$

and we denote respectively by $\widehat{\vartheta}^i_m(s) \subset \vartheta^i_m(s)$ and by $\widehat{\vartheta}_m(s) \subset \vartheta_m(s)$ their associated subsets of s-truncations.

Remark 9.1.1. By formula (9.1.3), if one introduces the decomposition

$$\widehat{\vartheta}^i_m(s) = \widehat{\vartheta}^i_m(1) \times \widehat{\vartheta}^i_m(s, 2) \tag{9.1.5}$$

as in (6.2.20), the space $\widehat{\vartheta}^i_m(1)$ is compact and $\widehat{\vartheta}^i_m(s, 2) \equiv \vartheta^i_m(s, 2)$.

Definition 9.1.2. In section 6.2.3 (see formula (6.2.23)), we defined the set $W^1(r, m)$ as

$$W^1(r, m) := \mathscr{F}^1(\mathcal{P}(r, m) \times \mathbb{R}^{m-1}) \,, \tag{9.1.6}$$

where the function $\mathscr{F}^1$ was defined in (6.2.21). Similarly, for any $j \in \{2, \ldots, m\}$, taking Remark 6.2.4 into account, we had set

$$W^j(r, m) := \mathscr{F}^j(\mathcal{P}(r, m) \times \mathbb{R}^{m-1}) \,. \tag{9.1.7}$$

Now, by (9.1.3), for any $i \in \{1, \ldots, m\}$ it makes sense to define also

$$\widehat{W}^i(r, m) := \mathscr{F}^i(\mathcal{P}(r, m) \times \overline{B}^{m-1}(0, \mathtt{M}(r, n, m, 1))) \,. \tag{9.1.8}$$

We remind that, due to Definition 9.1.1 and to Remark 6.2.3, for any given $i \in \{1, \ldots, m\}$ there exists a polynomial bijection $\mathfrak{U}^i$ between $\mathcal{P}(r, m) \times \widehat{\vartheta}^i_m(1)$ and $\widehat{W}^i(r, m)$: one is free to work either in the standard coordinates of $(p_\mu, \mathtt{a}) \in \mathbb{R}^M \times \mathbb{R}^{(m-1)s}$, with $M := \dim \mathcal{P}(r, m)$, or in the adapted coordinates of $(\mathtt{p}_\mu, \mathtt{a}) \in \widehat{W}^i(r, m)$.

By the arguments above, without any loss of generality, for any fixed $i \in \{1, \ldots, m\}$ it is sufficient to consider the set of those polynomials $P \in \mathcal{P}(r, m)$ verifying the s-vanishing condition on the s-jet $\mathcal{J}_{s,\gamma} \in \widehat{\vartheta}^i_m(s)$ of some curve $\gamma \in \widehat{\Theta}^i_m$. Namely, following (6.2.11) and (9.1.1), for $i = 1, \ldots, m$ we introduce the semi-algebraic sets

$$\begin{aligned}
\widehat{Z}^i(r, s, m) :=& \{(P, \widehat{\mathcal{J}}_{s,\gamma}) \in \mathcal{P}(r, m) \times \widehat{\vartheta}^i_m(s)\} | (P, \widehat{\mathcal{J}}_{s,\gamma}) \text{ satisfies} \\
& \qquad q^i_{\ell\alpha} \circ \Phi^i(P, \widehat{\mathcal{J}}_{s,\gamma}) = 0 \text{ for all } \ell \in \{1, ..., m\}, \alpha \in \{0, ..., s\}\} \\
\widehat{Z}(r, s, m) :=& \bigcup_{i=1}^{m} \widehat{Z}^i(r, s, m) \\
\widehat{\sigma}^i(r, s, m) :=& \Pi_{\mathcal{P}(r,m)} \widehat{Z}^i(r, s, m) \quad , \qquad \widehat{\sigma}(r, s, m) = \bigcup_{i=1}^{m} \widehat{\sigma}^i(r, s, m) \\
\widehat{\Sigma}(r, s, m) :=& \bigcup_{i=1}^{n} \widehat{\Sigma}^i(r, s, m) := \bigcup_{i=1}^{n} \text{closure}\left(\widehat{\sigma}^i(r, s, m)\right)
\end{aligned} \tag{9.1.9}$$

and, of course, we have

$$\widehat{Z}^i(r,s,m) \subset Z^i(r,s,m) \quad , \qquad \widehat{\sigma}^i(r,s,m) \subset \sigma^i(r,s,m) \quad \forall \ i \in \{1,\dots,m\}. \tag{9.1.10}$$

For further convenience, we also state the following simple

Lemma 9.1.1. *$\widehat{\sigma}(r,s,m)$ has codimension $s+m = \text{codim}\,\sigma(r,s,m)$ in $\mathcal{P}(r,m)$.*

Proof. By the third line of (9.1.9) and by the second inclusion in (9.1.10), it suffices to prove the statement for $\widehat{\sigma}^1(r,s,m)$. By the first line of (9.1.9) and by Definition 9.1.1, one has

$$\widehat{Z}^1(r,s,m) = Z^1(r,s,m) \bigcap \left\{ (P, \mathcal{J}_{s,\gamma}) \in \mathcal{P}(r,m) \times \widehat{\vartheta}^1_m(s) \right\} . \tag{9.1.11}$$

As it was shown in Corollary 6.2.1, the Jacobian associated to the equalities determining $Z^1(r,s,m)$ has full rank $ms+m$. Namely, by the discussion below expression (6.2.60), in the adapted coordinates of section 6.2.3, such a Jacobian has non-zero pivots corresponding to the derivatives w.r.t. the coefficients of the polynomial $\mathtt{P}_\mathtt{a}$ associated to multi-indices in the family (6.2.35). As we had shown in the proof of Corollary 6.2.1, this fact and the Implicit Function Theorem imply that for any pair $(P, \mathcal{J}_{s,\gamma})$ belonging to $Z^1(r,s,m)$, one can express $ms+m$ coefficients of P as implicit functions of the other coefficients of P and of the $(m-1)s$ parameters of $\mathcal{J}_{s,\gamma}$. This was the argument that led to estimate codim $\sigma^1(r,s,m) = s+m$ in Corollary 6.2.1. The thesis follows by putting together this argument with formulas (9.1.9)-(9.1.11) and with the fact that $\dim \vartheta^1_m(s) = \dim \widehat{\vartheta}^1_m(s)$. □

9.2 Partition of $\mathcal{P}(r,m)$ and $\widehat{W}^1(r,m)$

Let $r,m \geq 2$ be two integers. In this paragraph, we introduce a partition of the spaces $\mathcal{P}(r,m)$ and $\widehat{W}^1(r,m)$ which will turn out to be useful in the sequel. In order to do this, we first need to introduce a family of multi-indices.

9.2.1 A family of multi-indices

For $b,c \in \{2,\dots,m\}$, $b \leq c$ [5], we set

$$\varpi(b,c) := \mu = (\mu_1,\dots,\mu_m) \in \mathbb{N}^m \mid \mu_1 = 0 \,, \quad \mu_j = \delta_{jb} + \delta_{jc} \quad \forall j \in \{2,\dots,m\} \,. \tag{9.2.1}$$

[5] We have set $b \leq c$ in (9.2.1) only for convenience, in order not to have two indices b,c corresponding to the same multi-index $\mu \in \mathbb{N}^m$. Infact, it is clear that if we eliminate this constraint we have $\varpi(b,c) = \varpi(c,b)$ for all $b,c \in \{2,\dots,m\}$.

Comparing (9.2.1) with the sub-family $\nu(\mathtt{i},1)$ defined in (6.2.35), it is plain to check that one has the disjoint union

$$\left(\bigcup_{\substack{b,c=2\\ b\le c}}^{m}\{\varpi(b,c)\}\right)\bigsqcup\left(\bigcup_{\mathtt{i}=1}^{m}\{\nu(\mathtt{i},1)\}\right)=\{\mu\in\mathbb{N}^m \mid |\mu|=2\}\,. \tag{9.2.2}$$

Moreover, we have the following

Lemma 9.2.1. *For any polynomial $P\in\mathcal{P}(r,m)$, the coefficients p_μ associated to the multi-indices μ belonging to the family (9.2.1) are invariant under the transformations of paragraph 6.2.3. Namely, using the notations in (6.2.16), for any given $\mathtt{a}\in\widehat{\vartheta}^1_m(1)$ one has*

$$\mathtt{p}_{\varpi(b,c)}=p_{\varpi(b,c)}\qquad \textit{for all } b,c\in\{2,\dots,m\}\quad,\qquad b\le c\,.$$

Proof. We indicate by $A_1,\dots,A_m$ the orthonormal basis of $\mathbb{R}^m$ associated to the coordinates $x_1,\dots,x_m$ on which polynomials in $\mathcal{P}(r,m)$ depend.

For any $\mathtt{a}:=(a_{21},\dots,a_{m1})\in\widehat{\vartheta}^1_m(1)$, we also denote by

$$v_{\mathtt{a}}:=A_1+a_{21}A_2+a_{31}A_3+\ldots+a_{m1}A_m\,,\quad u_2:=A_2\,,\quad\ldots,\quad u_m:=A_m\,, \tag{9.2.3}$$

the basis associated to the adapted variables defined in Section 6.2.3 (see (6.2.13)), namely

$$\mathtt{y}_1:=x_1\quad,\qquad \mathtt{y}_2=\mathtt{y}_2(\mathtt{a}):=x_2-a_{21}\,x_1\quad\ldots\quad \mathtt{y}_m=\mathtt{y}_m(\mathtt{a}):=x_m-a_{m1}\,x_1\,. \tag{9.2.4}$$

By (6.2.16), (6.2.35) and (9.2.2), the quadratic terms of the transformed polynomial $\mathtt{P}_{\mathtt{a}}$ read

$$\begin{cases}\mathtt{p}_{\nu(1,1)}\mathtt{y}_1^2=\mathtt{p}_{\nu(1,1)}x_1^2\\ \mathtt{p}_{\nu(\ell,1)}\mathtt{y}_1\mathtt{y}_\ell=\mathtt{p}_{\nu(\ell,1)}x_1(x_\ell-a_{\ell1}x_1)\quad \ell=2,\dots,m\\ \mathtt{p}_{\varpi(j,\ell)}\mathtt{y}_j\mathtt{y}_\ell=\mathtt{p}_{\varpi(j,\ell)}(x_j-a_{j1}x_1)(x_\ell-a_{\ell1}x_1)\quad j,\ell=2,\dots,m\,,\quad j\le\ell\,.\end{cases} \tag{9.2.5}$$

By expression (9.2.5), we infer that - in the original variables $x_1,\dots,x_m$ - for any $j,\ell\in\{2,\dots,m\}$, $j\le\ell$, the coefficient associated to the monomial x_jx_ℓ is $\mathtt{p}_{\varpi(j,\ell)}$, that is $p_{\varpi(j,\ell)}=\mathtt{p}_{\varpi(j,\ell)}$.

□

9.2.2 Partition

For any $P\in\mathcal{P}(r,m)$, we set

$$\mathbb{H}^1(P):=\begin{pmatrix}2p_{\varpi(2,2)} & p_{\varpi(2,3)} & \cdots & p_{\varpi(2,m)}\\ p_{\varpi(2,3)} & 2p_{\varpi(3,3)} & \cdots & p_{\varpi(3,m)}\\ \cdots & \cdots & \ddots & \cdots\\ p_{\varpi(2,m)} & p_{\varpi(3,m)} & \cdots & 2p_{\varpi(m,m)}\end{pmatrix} \tag{9.2.6}$$

and

$$\mathsf{S}_1^1(r,m) := \{P \in \mathcal{P}(r,m)|\ \det \mathbb{H}^1(P) \neq 0\} . \tag{9.2.7}$$

Remark 9.2.1. Matrix $\mathbb{H}^1(P)$ is invariant under the transformations of section 6.2.3. Namely, by Lemma 9.2.1, we have

$$\mathbb{H}^1(\mathsf{P}_\mathsf{a}) := \begin{pmatrix} 2\mathsf{p}_{\varpi(2,2)} & \mathsf{p}_{\varpi(2,3)} & \cdots & \mathsf{p}_{\varpi(2,m)} \\ \mathsf{p}_{\varpi(2,3)} & 2\mathsf{p}_{\varpi(3,3)} & \cdots & \mathsf{p}_{\varpi(3,m)} \\ \cdots & \cdots & \ddots & \cdots \\ \mathsf{p}_{\varpi(2,m)} & \mathsf{p}_{\varpi(3,m)} & \cdots & 2\mathsf{p}_{\varpi(m,m)} \end{pmatrix} = \mathbb{H}^1(P) . \tag{9.2.8}$$

We also define

$$\mathsf{S}_2^1(r,m) := \mathcal{P}(r,m)\backslash \mathsf{S}_1^1(r,m) = \{P \in \mathcal{P}(r,m)|\ \det \mathbb{H}^1(P) = 0\} , \tag{9.2.9}$$

so that we can write the disjoint union

$$\mathcal{P}(r,m) = \mathsf{S}_1^1(r,m) \bigsqcup \mathsf{S}_2^1(r,m) . \tag{9.2.10}$$

We now consider the images of $\mathsf{S}_1^1(r,m)$ and $\mathsf{S}_2^1(r,m)$ through the transformation $\mathfrak{U}^1$ defined in Remark 6.2.3, namely

$$\begin{aligned} \mathcal{S}_1^1(r,m) &:= \mathfrak{U}^1\left(\mathsf{S}_1^1(r,m) \times \hat{\vartheta}_m^1(1)\right) = \left\{(\mathsf{P}_\mathsf{a},\mathsf{a}) \in \widehat{W}^1(r,m)|\ \det \mathbb{H}^1(\mathsf{P}_\mathsf{a}) \neq 0\right\} \\ \mathcal{S}_2^1(r,m) &:= \mathfrak{U}^1\left(\mathsf{S}_2^1(r,m) \times \hat{\vartheta}_m^1(1)\right) = \left\{(\mathsf{P}_\mathsf{a},\mathsf{a}) \in \widehat{W}^1(r,m)\ |\ \det \mathbb{H}^1(\mathsf{P}_\mathsf{a}) = 0\right\} . \end{aligned} \tag{9.2.11}$$

By (9.2.11), we have the partition

$$\widehat{W}^1(r,m) = \mathcal{S}_1^1(r,m) \bigsqcup \mathcal{S}_2^1(r,m) . \tag{9.2.12}$$

Remark 9.2.2. It is clear that the above partition can be implemented also in case one considers adapted variables $(\mathsf{P}_\mathsf{a},\mathsf{a}) \in \widehat{W}^i(r,m)$, with $i \in \{2,\dots,m\}$. By suitably modifying the family of indices in (9.2.1), as well as by introducing an adapted matrix $\mathbb{H}^i(P) = \mathbb{H}^i(P)$, it is possible to define sets $\mathsf{S}_1^i, \mathsf{S}_2^i$ whose disjoint union yields $\mathcal{P}(r,m)$ and sets $\mathcal{S}_1^i, \mathcal{S}_2^i$ whose disjoint union yields $\widehat{W}^i(r,m)$. However, the underlying reasonings are not conceptually different from the ones we did above, therefore we omit them in order not to burden the exposition.

9.3 Two important results

Consider three integers $r, m \geq 2$ and $s \in \{1,\dots,r-1\}$. The two results below are the cornerstones of the proof of Theorems C1-C2-C3.

Theorem 9.3.1. *In case $r \geq 2, s = 1$, for any $i \in \{1, \dots, m\}$ the semi-algebraic sets $\widehat{\sigma}^i(r, 1, m)$, and $\widehat{\sigma}(r, 1, m)$ are closed in $\mathcal{P}(r, m)$, that is, by formulas (9.1.9),*

$$\widehat{\sigma}^i(r, 1, m) = \widehat{\Sigma}^i(r, 1, m) \quad \forall\, i \in \{1, \dots, m\} \quad , \qquad \widehat{\sigma}(r, 1, m) = \widehat{\Sigma}(r, 1, m) \; .$$

Moreover, for any $i \in \{1, \dots, m\}$, taking the definition of transformation Υ^i into account (see (6.2.25)), the set $\Pi_{\widehat{W}^i(r,m)} \Upsilon^i(\widehat{Z}^i(r, 1, m))$ is closed in $\widehat{W}^i(r, m)$, and its form can be explicitly computed.

Theorem 9.3.2. *For any given values of $r \geq 3$, $s \geq 2$, and $i \in \{1, \dots, m\}$ there exist two semi-algebraic subsets of $\mathcal{P}(r, m)$*

$$X_1^i(r, s, m) \subset \mathsf{S}_1^i(r, m) \; , \;\; X_2^i(r, s, m) \subset \mathsf{S}_2^i(r, m) \; , \tag{9.3.1}$$

and two semi-algebraic subsets of $\widehat{W}^i(r, m)$

$$Y_1^i(r, s, m) \subset \mathcal{S}_1^i(r, m) \; , \;\; Y_2^i(r, s, m) \subset \mathcal{S}_2^i(r, m) \; , \tag{9.3.2}$$

satisfying the following properties:

1. *for $j \in \{1, 2\}$, one has*

$$X_j^i(r, s, m) = \Pi_{\mathcal{P}(r,m)} \left((\mathfrak{U}^1)^{-1} (Y_j^i(r, s, m)) \right) \; ,$$

   where $\mathfrak{U}^1$ was defined in Remark 6.2.3;

2. *$Y_1^i(r, s, m)$ is closed in $\mathcal{S}_1^i(r, m)$ for the induced topology;*
3. *$X_1^i(r, s, m)$ is closed in $\mathsf{S}_1^i(r, m)$ for the induced topology;*
4. *one has the partition $\widehat{\sigma}^i(r, s, m) = X_1^i(r, s, m) \bigsqcup X_2^i(r, s, m)$;*
5. *the form of $Y_1^i(r, s, m)$ can be explicitly computed by the means of an algorithm involving only linear operations.*

The rest of this section is devoted to the proof of the above results.

We will only prove Theorems 9.3.1-9.3.2 in the case $i = 1$, as the other cases are simple generalizations.

9.3.1 Strategy of proof of Theorems 9.3.1-9.3.2.

We have seen in section 6 that the equations determining $Z^1(r, s, m)$, can be written in the adapted coordinates introduced in section 6.2.3. Namely, by recalling the functions

$$\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) : W^1(r, m) \times \mathbb{R}^{(m-1)(s-1)} \longrightarrow \mathbb{R} \quad , \qquad \mathtt{i} \in \{1, \dots, m\} \quad \alpha \in \{0, \dots, s\} \, , \tag{9.3.3}$$

presented in (6.2.32)-(6.2.33), Lemma 6.2.3 ensures that $Z^1(r,s,m)$ is the image through the inverse of the transformation Υ^1 in (6.2.26) of the zero set

$$\bigcap_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}} \{(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) \in W^1(r,m)\times\mathbb{R}^{(m-1)(s-1)} | \mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = 0\}\ . \tag{9.3.4}$$

Then, if we indicate by $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})$ the restriction of $\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}})$ to the subset $\widehat{W}^1(r,m)\times\vartheta^1_m(s,2)$, and if we denote by $\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)$ the zero set of the non-linear maps $\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}$, it is clear by the discussion at paragraph 9.1, in particular by (9.1.11), that

$$\widehat{Z}^1(r,s,m) := (\Upsilon^1)^{-1}\left(\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)\right) \tag{9.3.5}$$

that is

$$\Upsilon^1(\widehat{Z}^1(r,s,m)) = \mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)\ . \tag{9.3.6}$$

Taking (9.3.6) and the third line of (9.1.9) into account, the key idea behind the proof of Theorems 9.3.1-9.3.2 consists in understanding under which conditions the set $\Upsilon^1(\widehat{Z}^1(r,s,m))$ admits a closed projection onto $\widehat{W}^1(r,m)$.

Namely, we assume the existence of a semi-algebraic subset of polynomials

$$\mathsf{S}^1_0(r,s,m)\subset\mathcal{P}(r,m) \tag{9.3.7}$$

for which it is possible to extrapolate linearly from the $ms+m$ equations in (9.3.6) the $(m-1)s$ parameters of $\vartheta^1_m(s,2)$ as explicit algebraic functions of the parameters of $\widehat{W}^1(r,m)$.

Remark 9.3.1. The fact of being able to reduce linearly the coefficients of $\vartheta^1_m(s,2)$ from (9.3.6) may in principle depend also on the values of the parameters $\mathtt{a}\in\widehat{\vartheta}^1_m(1)$, so that the set in (9.3.7) should depend also on $\mathtt{a}$. Moreover, for the moment we have no elements that allow us to establish that a semi-algebraic set $\mathsf{S}^1_0(r,s,m)$ with the required properties exists. In order to understand the sequel, we stress that here we simply make a working hypothesis on the existence of a semi-algebraic subset $\mathsf{S}^1_0(r,s,m)$ on which it is possible to reduce linearly the parameters of $\vartheta^1_m(s,2)$ from (9.3.6) independently of the value of the parameter $\mathtt{a}\in\widehat{\vartheta}^1_m(1)$. The validity of this hypothesis will be verified in the sequel.

Then, we introduce

$$\mathsf{A}(r,s,m) :=\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)\bigcap\Upsilon^1(\mathsf{S}^1_0(r,s,m)\times\widehat{\vartheta}^1_m(s))\ , \tag{9.3.8}$$

and we remind that, by Remark 6.2.3,

$$\Upsilon^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(s)) = \mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))\times\vartheta_m^1(s,2)\ . \tag{9.3.9}$$

We can now state

Lemma 9.3.1. *With the setting above, the two following properties hold:*

1. *the projection*
$$\mathsf{A}'(r,s,m) := \Pi_{\widehat{W}^1(r,m)}\mathsf{A}(r,s,m)$$
is a semi-algebraic subset which is closed in $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\vartheta_m^1(1))$ for the induced topology;

2. *the projection $\Pi_{\mathcal{P}(r,m)}\left((\mathfrak{U}^1)^{-1}(\mathsf{A}'(r,s,m))\right)$ is a semi-algebraic subset which is closed in $\mathsf{S}_0^1(r,s,m)$ for the induced topology.*

Proof.

Step 1. By construction, at any point of $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))\times\vartheta_m^1(s,2)$ the $(m-1)(s-1)$ parameters of the space $\vartheta_m^1(s,2)$ can be reduced explicitly from the $ms+m$ algebraic equations in (9.3.6) with the help of linear algorithms. This means that $\mathsf{A}(r,s,m)$ is determined by a system of $ms+m-(m-1)(s-1)=s+2m-1$ algebraic equations involving only the coordinates of $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))$, and of $(m-1)(s-1)$ algebraic equations that parametrize the coefficients of $\vartheta_m^1(s,2)$ as functions of the points in $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))$. In other words, $\mathsf{A}(r,s,m)$ has the form of a graph of the type

$$\mathsf{A}(r,s,m) = \mathsf{A}'(r,s,m)\times\overline{\theta}(r,s,m)\ , \tag{9.3.10}$$

where $\mathsf{A}'(r,s,m) := \Pi_{\widehat{W}^1(r,m)}\mathsf{A}(r,s,m)$ is a closed subset of $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))$ for the induced topology determined by algebraic equations involving the coordinates of elements in $\mathfrak{U}^1(\mathsf{S}_0^1(r,s,m)\times\widehat{\vartheta}_m^1(1))$, and the points of $\overline{\theta}(r,s,m)\subset\vartheta_m^1(s,2)$ are parametrized by $\mathsf{A}'(r,s,m)$.

Moreover, by Remark 6.2.3 the function $\mathfrak{U}^1$ is polynomial, and we have assumed as a working hypothesis that $\mathsf{S}_0^1(r,s,m)$ is semi-algebraic in $\mathcal{P}(r,m)$ (see Remark 9.3.1). Therefore, by (9.3.8)-(9.3.9) the set $\mathsf{A}(r,s,m)$ is semi-algebraic in $\widehat{W}^1(r,m)\times\vartheta_m^1(s,2)$, and $\mathsf{A}'(r,s,m)$ is semi-algebraic in $\widehat{W}^1(r,m)$ by the Theorem of Tarski and Seidenberg A.1.1.

Step 2. Since the invertible transformation $\mathfrak{U}^1$ defined in Remark 6.2.3 is polynomial, due to Step 1 and to continuity we have that the inverse image

$$(\mathfrak{U}^1)^{-1}(\mathsf{A}'(r,s,m))\subset\mathsf{S}_0^1(r,m)\times\widehat{\vartheta}_m^1(1) \tag{9.3.11}$$

is closed in $\mathsf{S}_0^1(r,m)\times\widehat{\vartheta}_m^1(1)$ for the induced topology. Finally - taking into account the fact that, as we have already pointed out in Remark 9.1.1, $\widehat{\vartheta}_m^1(1)$ is compact - Lemma C.2.1 ensures that the projection

$$\Pi_{\mathcal{P}(r,m)}\left((\mathfrak{U}^1)^{-1}(\mathsf{A}'(r,s,m))\right) \tag{9.3.12}$$

is closed in $\mathsf{S}_0^1(r,s,m)$ for the induced topology.

The semi-algebraicness of the projection in (9.3.12) is a consequence of the semi-algebraicness of $\mathsf{A}'(r,s,m)$ demonstrated at Step 1, of the fact that $\mathfrak{U}^1$ is polynomial, and of the Theorem of Tarski and Seidenberg.

□

Corollary 9.3.1. *In case $\mathsf{S}_0(r,s,m) \equiv \mathcal{P}(r,m)$, the subsets*

$$\mathsf{A}'(r,s,m) \quad , \qquad \Pi_{\mathcal{P}(r,m)}\left((\mathfrak{U}^1)^{-1}(\mathsf{A}'(r,s,m))\right) = \widehat{\sigma}^1(r,s,m)\,.$$

are both closed in $\widehat{W}^1(r,m)$ and $\mathcal{P}(r,m)$, respectively.

In particular, one has $\widehat{\sigma}^1(r,s,m) = \widehat{\Sigma}^1(r,s,m)$.

Proof. It is immediate by (9.3.8) and (9.3.6) that if $\mathsf{S}_0(r,s,m) \equiv \mathcal{P}(r,m)$ then

$$\mathsf{A}(r,s,m) := \mathscr{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right) = \Upsilon^1(\widehat{Z}^1(r,s,m))\,.$$

The thesis follows by considering the above expression together with (9.1.9) and Lemma 9.3.1. □

Corollary 9.3.2. *The same result of Lemma 9.3.1 holds if one considers a semi-algebraic subset $\widetilde{\mathsf{S}}_0^1(r,m) \subset \mathsf{S}_0^1(r,m)$. One just has to replace $\mathsf{S}_0^1(r,m)$ with $\widetilde{\mathsf{S}}_0^1(r,m)$ in the statement.*

Proof. One just needs to follow the same steps in the proof of Lemma 9.3.1 and to consider $\widetilde{\mathsf{S}}_0^1(r,m)$ instead of $\mathsf{S}_0^1(r,m)$. □

By the above reasonings, the goal in the next paragraphs is to study in depth the form of the equations in (9.3.4)-(9.3.6) in order to see in which cases the parameters of $\vartheta_m^1(s,2)$ can be reduced by the means of linear algorithms. For $\mathtt{i} \in \{1,\dots,m\}$, and $\alpha \in \{0,\dots,s\}$, the explicit expressions of the functions $\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}})$ are given in formulas (6.2.39)-(6.2.40) of Lemma 6.2.3.

As we shall see, the linear reduction of the parameters of $\vartheta_m^1(s,2)$ is always possible in case $r \geq 2$, $s = 1$, or $r \geq 3$, $s = 2$, in other words in this regime one has $\mathsf{S}_0^1(r,s,m) \equiv \mathcal{P}(r,m)$.

For $r \geq 4$, $s \geq 3$, the linear reduction of $\vartheta_m^1(s,2)$ is possible for polynomials belonging to set $\mathsf{S}_1^1(r,m)$ in (9.2.7), that is $\mathsf{S}_0^1(r,s,m) \supset \mathsf{S}_1^1(r,m)$.

9.3.2 Proof of Theorem 9.3.1.

When $s = 1$, then in (6.2.39)-(6.2.40) one must consider $\alpha \in \{0,1\}$. We observe that

$$\forall\, \alpha \in \{0,1\}\,, \quad \forall \mathtt{i} = 1,\dots,m \qquad \widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0 \quad \Longleftrightarrow \quad \mathtt{p}_{\nu(\mathtt{i},\alpha)} = 0\,, \tag{9.3.13}$$

where the family of multi-indices $\nu(\mathtt{i},\alpha)$ was defined in (6.2.35). As we see, no parameters belonging to the space $\vartheta^1_m(s,2)$ appear in (9.3.13), so that with the notations of paragraph 9.3.1, we have $\mathsf{S}^1_0(r,1,m) \equiv \mathcal{P}(r,m)$ and (9.3.13) corresponds to the explicit expression of the closed set $\mathsf{A}'(r,1,m)$. Moreover, by Corollary 9.3.1, $\widehat{\sigma}^1(r,1,m)$ is closed in $\mathcal{P}(r,m)$ and, by (9.1.9), one has $\widehat{\sigma}^1(r,1,m) = \widehat{\Sigma}^1(r,1,m)$.

This proves the statement of Theorem 9.3.1 for $r \geq 2$, $s = 1$.

9.3.3 Explicit form of the equations (case $r \geq 3$, $s \geq 2$)

The goal of this paragraph is to give a more explicit expression of $\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,1\}}}\right)$, in case $r \geq 3$, $s \geq 2$.

Remark 9.3.2. We remind that equations (6.2.39)-(6.2.40) are recursive w.r.t. the parameters of the curve γ. Namely, for any given integer $\beta \in \{2,\dots,s\}$, the coefficients of order β belonging to the space $\vartheta^1_m(s,2)$ - that is $a_{22},\dots,a_{2\beta},\dots,a_{m2},\dots,a_{m\beta}$ - appear in equations (6.2.39)-(6.2.40) only for $\alpha \geq \beta$.

For any polynomial $P \in \mathcal{P}(r,m)$ and any curve $\gamma \in \widehat{\Theta}^1_m$, we indicate by $\mathtt{P}^{(>2)}_{\mathtt{a}}$ the associated polynomial $\mathtt{P}_{\mathtt{a}}$ written in the adapted coordinates for γ (see paragraph 6.2.3) deprived of its monomials of degree less or equal than two [6]. Also, if $\mathcal{J}_{s,\gamma} \in \widehat{\vartheta}^1_m(1)$ is the s-truncation of γ, for any given $\alpha \in \{2,\dots,s\}$ we denote by

$$\mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}(t) := \mathtt{J}_{s,\gamma,\mathtt{a}} - \begin{pmatrix} 0 \\ \sum_{i=\alpha}^{s} a_{2i}t^i \\ \dots \\ \sum_{j=\alpha}^{s} a_{mj}t^j \end{pmatrix}$$

its truncation at order $\alpha - 1$ written in the adapted coordinates for γ.

Remark 9.3.3. We observe that, for $\alpha = 2$, $\mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<2)}(t)$ reduces to the line $(t,0,\dots,0)$, since with the exception of the parametrizing coordinate, the components of $\mathtt{J}_{s,\gamma,\mathtt{a}}$ start at order two in t (see paragraph 6.2.3).

With this setting, we have

Lemma 9.3.2. *For any polynomial $P \in \mathcal{P}(r,m)$, there exists a linear change of coordinates $\mathfrak{D} = \mathfrak{D}(P) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ such that for any given $\alpha \in \{2,\dots,s\}$, and for $\mathtt{i} = 1,\dots,m$, the algebraic equations $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}},\mathtt{a},\mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ in (6.2.39)-(6.2.40) take the*

[6] One has $\mathtt{P}^{(>2)}_{\mathtt{a}} \neq 0$ since we are considering the case of polynomials having degree $r \geq 3$.

form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2\mathtt{p}'_{\varpi(2,2)} & 0 & \dots & 0 \\ 0 & 0 & 2\mathtt{p}'_{\varpi(3,3)} & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2\mathtt{p}'_{\varpi(m,m)} \end{pmatrix} \begin{pmatrix} 0 \\ a'_{2\alpha} \\ a'_{3\alpha} \\ \dots \\ a'_{m\alpha} \end{pmatrix} + \mathfrak{D} \begin{pmatrix} \widehat{\mathtt{Q}}_{1\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \widehat{\mathtt{Q}}_{2\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \widehat{\mathtt{Q}}_{3\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \dots \\ \widehat{\mathtt{Q}}_{m\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \end{pmatrix} = 0\,. \tag{9.3.14}$$

Proof. Step 1. We firstly claim that equations (6.2.39)-(6.2.40) can be but into the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2\mathtt{p}_{\varpi(2,2)} & \mathtt{p}_{\varpi(2,3)} & \dots & \mathtt{p}_{\varpi(2,m)} \\ 0 & \mathtt{p}_{\varpi(2,3)} & 2\mathtt{p}_{\varpi(3,3)} & \dots & \mathtt{p}_{\varpi(3,m)} \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \mathtt{p}_{\varpi(2,m)} & \mathtt{p}_{\varpi(3,m)} & \dots & 2\mathtt{p}_{\varpi(m,m)} \end{pmatrix} \begin{pmatrix} 0 \\ a_{2\alpha} \\ a_{3\alpha} \\ \dots \\ a_{m\alpha} \end{pmatrix} + \begin{pmatrix} \widehat{\mathtt{Q}}_{1\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \widehat{\mathtt{Q}}_{2\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \widehat{\mathtt{Q}}_{3\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \\ \dots \\ \widehat{\mathtt{Q}}_{m\alpha}\left(\mathtt{P}_{\mathtt{a}}^{(>2)}, \mathtt{J}_{s,\gamma,\mathtt{a}}{}^{(<\alpha)}\right) \end{pmatrix} = 0\,. \tag{9.3.15}$$

By (9.3.13), for any $\alpha \in \{2, \dots, s\}$, and for any $\mathtt{i}, \mathtt{j} \in \{1, \dots, m\}$, the monomials $\mathtt{p}_{\nu(\mathtt{j},0)}, \mathtt{p}_{\nu(\mathtt{j},1)}$ do not yield any contribution to equations $\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$. Therefore, taking (9.2.2) into account, the only monomials of order two which may appear in equation $\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ are those associated to the multi-indices $\varpi(b, c)$, with $b, c \in \{2, \dots, m\}$, $b \le c$.

Moreover, fixing the values of $\mathtt{i} \in \{1, \dots, m\}$ and $\alpha \in \{2, \dots, s\}$, by (6.2.39)-(6.2.40), the multi-indices $\mu \in \mathbb{N}^m$ contributing to equation $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ are those for which the set $\mathcal{G}_m(\widetilde{\mu}(\mathtt{i}), \alpha)$ in (6.2.37) is non-empty. This amounts to requiring that the components of the vector $(k_{22}, \dots, k_{2\alpha}, \dots, k_{m2}, \dots, k_{m\alpha}) \in \mathbb{N}^{(m-1)(\alpha-1)}$ appearing in formulas (6.2.39)-(6.2.40) satisfy

$$\sum_{i=2}^{\alpha} k_{ji} = \widetilde{\mu}_j(\mathtt{i}) \quad \forall j \in \{2, \dots, m\} \quad , \qquad \widetilde{\mu}_1(\mathtt{i}) + \sum_{j=2}^{m}\sum_{i=2}^{\alpha} i\, k_{ji} = \alpha\,. \tag{9.3.16}$$

In particular, for fixed $\mathtt{i} \in \{1, \dots, m\}$, $\alpha \in \{2, \dots, s\}$ and $\ell \in \{2, \dots, m\}$, if we look at the monomials containing the coefficient $a_{\ell\alpha}$ in equation $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ - that is, at the form of the terms for which $k_{\ell\alpha} \neq 0$ in (6.2.39)-(6.2.40) - by (9.3.16) we must have

$$\widetilde{\mu}_1(\mathtt{i}) = 0 \quad , \qquad k_{ji} = \delta_{i\alpha}\delta_{j\ell} \quad , \qquad \widetilde{\mu}_j(\mathtt{i}) = \delta_{j\ell} \quad , \qquad j \in \{2, \dots, m\}\,. \tag{9.3.17}$$

Firstly, expression (9.3.17) implies $|\mu| = 2$. Secondly, as we said above, the only multi-indices of length $|\mu| = 2$ which may yield a contribution to $\mathtt{Q}_{\mathtt{i}\alpha}(\mathtt{P}_{\mathtt{a}}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ are those belonging to the family $\{\varpi(b, c)\}_{b,c \in \{2,\dots,m\} b \le c}$ in (9.2.1). Therefore, we have two cases.

Case $\mathtt{i} = 1$. If $\mu \in \{\varpi(b,\ell)\}_{b,\ell\in\{2,\dots,m\} b\le\ell}$ then $\mu_1 = 0$, and all terms in formula (6.2.39) which are associated to these indices are multiplied by zero. Hence, the coefficients $a_{\ell\alpha}$ do not appear in $\widehat{\mathtt{Q}}_{1\alpha}(\mathtt{P}_\mathtt{a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$, nor do any of the monomials of order two in $\mathtt{P}_\mathtt{a}$. This, together with Remark 9.3.2 proves the claim for $\mathtt{i} = 1$ (the first line of (9.3.15)).

Case $\mathtt{i} \in \{2,\dots,m\}$. Taking (9.3.17) into account, for any given $\ell \in \{2,\dots,m\}$ one has that

1. if $\mathtt{i} \le \ell$, the only term to which the coefficient $a_{\ell\alpha}$ is associated in equation $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_\mathtt{a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ is the one corresponding to the multi-index $\mu \in \varpi(\mathtt{i},\ell)$, that is, by (6.2.40), the monomial $(1+\delta_{\ell\mathtt{i}})\mathtt{p}_{\varpi(\mathtt{i},\ell)}a_{\ell\alpha}$;

2. if $\mathtt{i} > \ell$, the term containing $a_{\ell\alpha}$ is the one associated to the multi-index $\mu \in \varpi(\ell,\mathtt{i})$, that is, by (6.2.40), $(1+\delta_{\ell\mathtt{i}})\mathtt{p}_{\varpi(\ell,\mathtt{i})}a_{\ell\alpha}$.

Conversely, if a monomial associated to an index $\varpi(\mathtt{i},\ell)$, with $\mathtt{i},\ell \in \{2,\dots,m\}$, $\mathtt{i} \le \ell$, appears in equations $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_\mathtt{a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$, then, by (9.2.1) and by (9.3.16), one must necessarily have

$$\begin{cases} \widetilde{\mu}_1(\mathtt{i}) + \sum_{j=2}^m \sum_{i=2}^\alpha i\, k_{ji} = \alpha \quad , \qquad \widetilde{\mu}_1(\mathtt{i}) = 0 \\ \sum_{i=2}^\alpha k_{ji} = \widetilde{\mu}_j(\mathtt{i}) := \delta_{j\ell} + \delta_{j\mathtt{i}} - \delta_{j\mathtt{i}} = \delta_{j\ell} \quad \forall j \in \{2,\dots,m\}\, , \end{cases} \tag{9.3.18}$$

which is true if and only if for some $v \in \{2,\dots,\alpha\}$ one has

$$\begin{cases} k_{ji} = \delta_{j\ell}\delta_{iv} \\ \sum_{j=2}^m \sum_{i=2}^\alpha i\, \delta_{j\ell}\delta_{iv} = \alpha \end{cases} \tag{9.3.19}$$

that is if and only if $k_{ji} = \delta_{j\ell}\delta_{i\alpha}$. One can check by formula (6.2.40) that this ensures that such a term must be of the form $(1+\delta_{\ell\mathtt{i}})\mathtt{p}_{\varpi(\mathtt{i},\ell)}a_{\ell\alpha}$. This reasoning, together with Remark 9.3.2, and with the fact that - as we showed at the beginning of the proof - no monomials of order two appear in equation $\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}(\mathtt{P}_\mathtt{a}, \mathtt{a}, \mathtt{J}_{s,\gamma,\mathtt{a}}) = 0$ other than those associated to the family (9.2.1), proves the claim for $\mathtt{i} \in \{2,\dots,m\}$ (lines $2,\dots,m$ of (9.3.15)).

Step 2. For any $P \in \mathcal{P}(r,m)$, taking Lemma 9.2.1 into account we indicate by $\mathbb{G}^1(P)$ the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 2p_{\varpi(2,2)} & \dots & p_{\varpi(2,m)} \\ 0 & \dots & \ddots & \dots \\ 0 & p_{\varpi(2,m)} & \dots & 2p_{\varpi(m,m)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 2\mathtt{p}_{\varpi(2,2)} & \dots & \mathtt{p}_{\varpi(2,m)} \\ 0 & \dots & \ddots & \dots \\ 0 & \mathtt{p}_{\varpi(2,m)} & \dots & 2\mathtt{p}_{\varpi(m,m)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{H}^1(P) \end{pmatrix} \tag{9.3.20}$$

appearing in (9.3.15). $\mathbb{G}^1(P)$ is symmetric, hence diagonalizable. Hence, for any $P \in \mathcal{P}(r,m)$, there exist a basis of eigenvectors, indicated by

$$v_\mathtt{a},\, u_2' = \mathtt{u}_2'(P), \dots,\, u_m' = u_m'(P)\, , \tag{9.3.21}$$

and a real $m\times m$ invertible matrix $\mathfrak{D} = \mathfrak{D}(P)$, such that equation (9.3.14) takes the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2\mathrm{p}'_{\varpi(2,2)} & 0 & \dots & 0 \\ 0 & 0 & 2\mathrm{p}'_{\varpi(3,3)} & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2\mathrm{p}'_{\varpi(m,m)} \end{pmatrix} \begin{pmatrix} 0 \\ a'_{2\alpha} \\ a'_{3\alpha} \\ \dots \\ a'_{m\alpha} \end{pmatrix} + \mathfrak{D} \begin{pmatrix} \mathrm{Q}_{1\alpha}\left(\mathrm{P}_{\mathrm{a}}^{(>2)}, \mathrm{J}_{s,\gamma,\mathrm{a}}{}^{(<\alpha)}\right) \\ \mathrm{Q}_{2\alpha}\left(\mathrm{P}_{\mathrm{a}}^{(>2)}, \mathrm{J}_{s,\gamma,\mathrm{a}}{}^{(<\alpha)}\right) \\ \mathrm{Q}_{3\alpha}\left(\mathrm{P}_{\mathrm{a}}^{(>2)}, \mathrm{J}_{s,\gamma,\mathrm{a}}{}^{(<\alpha)}\right) \\ \dots \\ \mathrm{Q}_{m\alpha}\left(\mathrm{P}_{\mathrm{a}}^{(>2)}, \mathrm{J}_{s,\gamma,\mathrm{a}}{}^{(<\alpha)}\right) \end{pmatrix} = 0\,, \tag{9.3.22}$$

where the primed quantities indicate that we are working in the new basis (9.3.21).

Remark 9.3.4. Comparing (9.2.3) with (9.3.21), we observe that the vector v_{a} was left unchanged. This is due to the fact that, by (9.3.20), v_{a} is already an eigenvector of $\mathbb{G}^1(P)$ (associated to a null eigenvalue).

□

9.3.4 Proof of Theorem 9.3.2

For $r \geq 3$, $s \geq 2$, we define

$$X_1^1(r,s,m) := \widehat{\sigma}^1(r,s,m) \bigcap \mathsf{S}_1^1(r,m)\,. \tag{9.3.23}$$

Firstly, we show that, if $P \in \mathsf{S}_1^1(r,m)$, then the parameters of the space $\vartheta_m^1(s,2)$ can be reduced iteratively from equation (9.3.14) for $\alpha \in \{2,\dots,s\}$.

When $\alpha = 2$ the second term at the l.h.s. of (9.3.14) does not depend on the parameters of $\vartheta_m^1(s,2)$ (see Remark 9.3.3), so that the coefficients $a_{22},\dots,a_{m2}$ can be reduced, as matrix $\mathbb{H}^1(P)$ is invertible by construction.

If, for $\alpha \in \{3,\dots,s\}$, we assume that the parameters $a_{j\beta}$, with $j \in \{2,\dots,m\}$ and $\beta \in \{2,\dots,\alpha-1\}$, have been reduced, then the first equation in (9.3.14) does not contain any new parameter, whereas the terms $a_{2\alpha},\dots,a_{m\alpha}$ can be found by inverting $\mathbb{H}^1(P)$ once again.

The above considerations and (9.3.23) imply that if $P \in X_1^1(r,s,m)$ then

1. the parameters of the space $\vartheta_m^1(s,2)$ can be reduced from the equations in (9.3.14) by the means of a recursive algorithm which only involves linear computations and the inversion of $\mathbb{H}^1(P)$;

2. there exists a truncation $\mathcal{J}_{s,\gamma} \in \vartheta_m^1(s)$ such that $\mathfrak{U}^1(P\times\mathcal{J}_{s,\gamma}) = (\mathrm{P}_{\mathrm{a}}, \mathrm{a}, \mathrm{J}_{s,\gamma,\mathrm{a}})$ solves (9.3.14).

Taking (9.3.8) and the above arguments into account, with the notations of section 9.3.1 we have that

1. $\mathsf{S}_1^1(r,m) \subset \mathsf{S}_0^1(r,s,m)$ for all $s \geq 2$;

2. by the previous point and by Corollary 9.3.2, the projection

$$Y_1^1(r,s,m) := \Pi_{\widehat{W}^1(r,m)}\left(\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)\bigcap \Upsilon^1(\mathsf{S}_1^1(r,m)\times\widehat{\vartheta}_m^1(s))\right)$$

is closed in $\mathfrak{U}^1(\mathsf{S}_1^1(r,m)\times\widehat{\vartheta}_m^1(1)) =: \mathscr{S}_1^1(r,m)$ for the induced topology and semi-algebraic in $\widehat{W}^1(r,m)$. Moreover, still by Corollary 9.3.2, taking into account (9.3.23) the projection

$$\Pi_{\mathcal{P}(r,m)}\left((\mathfrak{U}^1)^{-1}(Y_1^1(r,s,m))\right) = \widehat{\sigma}^1(r,s,m)\bigcap \mathsf{S}_1^1(r,m) =: X_1^1(r,s,m)$$

is a semi-algebraic subset of $\mathcal{P}(r,m)$, closed in $\mathsf{S}_1^1(r,m)$ for the induced topology;

3. as, due to (9.3.13), for $\alpha = 0, 1$ no parameters of $\vartheta_m^1(s,2)$ appear in the equations determining the set in (9.3.6), and as for any $P \in \mathsf{S}^1(r,m)$ the parameters of $\vartheta_m^1(s,2)$ can be reduced recursively from (9.3.14) by a linear algorithm when $2 \le \alpha \le s$, the form of $Y_1^1(r,s,m)$ can be obtained by performing solely linear operations.

The above arguments prove Theorem 9.3.2 once one sets

$$\begin{aligned} X_2^1(r,s,m) &:= \widehat{\sigma}^1(r,s,m)\backslash X_1^1(r,s,m)\\ Y_2^1(r,s,m) &:= \Pi_{\widehat{W}^1(r,m)}\left(\mathcal{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s\}}}\right)\right)\backslash Y_1^1(r,s,m)\,. \end{aligned} \tag{9.3.24}$$

Chapter 10

Proof of Theorems C1-C2-C3

We assume the notations of section 9. In this context, we are considering two positive integers $r \geq 2$, $n \geq 3$, a vector $\mathtt{s} := (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$, with $1 \leq s_i \leq r-1$ for all $i = 1, \dots, n-1$, and a function h of class C_b^{2r-1} around the origin, satisfying $\nabla h(0) \neq 0$.

Also, for any $m \in \{2, \dots, n-1\}$, we set $\mathscr{K}(r,n,m) := \mathtt{M}(r,n,m,1)$, where the constant $\mathtt{M}(r,n,m,1)$ was defined in Lemma 5.0.1 (see formula 5.0.1).

10.1 Proof of Theorem C1

Fix $m \in \{2, \dots, n-1\}$. Let Γ^m be a m-dimensional subspace belonging to the subset $\Lambda_0(h,m,n) \subset \mathtt{G}(m,n)$ introduced in Definition 4.2.1.

Taking Definition 9.1.3 into account, we consider a curve $\gamma \in \widehat{\Theta}_m$ whose image is contained in Γ^m. Without any loss of generality, up to changing the order of the vectors spanning Γ^m, we can suppose that γ is parametrized by the first coordinate, hence that $\gamma \in \widehat{\Theta}_m^1$. Following (9.1.2), we indicate by $\mathtt{a} = (a_{21}, \dots, a_{m1}) \in \overline{B}^{m-1}(0, \mathscr{K}(r,n,m))$ the linear Taylor coefficients of γ at the origin, and by $\mathcal{J}_{s,\gamma}$ its s-truncation (with $1 \leq s \leq r-1$).

We also indicate by $u_1, \dots, u_m \in \mathtt{U}(m,n)$ a orthonormal basis spanning Γ^m, and by $v := u_1 + \sum_{i=2}^m a_{i1} u_i, u_2 \dots, u_m$ the basis associated to the adapted coordinates for γ introduced in section 6.2.3. As we have already shown in (8.1.3), the Taylor polynomial $\mathtt{T}_0(h,r,n)$ restricted to Γ^m written in the adapted coordinates reads

$$\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m)(\mathtt{y}) = \sum_{\substack{\mu \in \mathbb{N}^m \\ 1 \leq |\mu| \leq r}} \frac{1}{\mu!} h_0^{|\mu|} \Big[\overbrace{v}^{\mu_1}, \overbrace{u_2}^{\mu_2}, \dots, \overbrace{u_m}^{\mu_m} \Big] \mathtt{y}_1^{\mu_1} \dots \mathtt{y}_m^{\mu_m}, \tag{10.1.1}$$

where we have used the notation introduced in formula (3.0.2). Moreover, as customary, $\mathcal{J}_{s,\gamma}$ reads $\mathtt{J}_{s,\gamma,\mathtt{a}}$ in the new coordinates.

By the arguments at paragraph 9.1, and by taking (6.2.25) and (9.1.9) into account, if we manage to prove that condition

$$(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{1,\gamma,\mathtt{a}}) \in \Upsilon^1(\widehat{Z}^1(r,1,m))$$

is never satisfied for any choice of the curve γ, which is equivalent - due to Theorem 9.3.1 for $s = 1$ - to condition

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \notin \widehat{\sigma}^1(r,1,m) = \widehat{\Sigma}^1(r,1,m)\ ,$$

then by the definitions in (7.1.1)-(7.1.2) we have

$$\mathtt{T}_0(h, r, n) \in \mathcal{P}(r,n) \backslash \mathcal{V}(r,1,m,n)$$

and the thesis follows by putting the expression above together with (7.3.1) and Theorem A.

By Lemma 6.2.3 and by (6.2.33), condition

$$(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{1,\gamma,\mathtt{a}}) \notin \Upsilon^1(\widehat{Z}^1(r,1,m))$$

is equivalent to asking that system

$$\mathtt{Q}_{i\alpha}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m}, r, m), \mathtt{a}, \mathtt{J}_{1,\gamma,\mathtt{a}}) = 0 \qquad \forall\, i \in \{1,\dots,m\}\ ,\ \ \forall \alpha \in \{0,1\}\ , \qquad (10.1.2)$$

has no solution. Then, by expressions (8.1.8)-(8.1.9), the system in (10.1.2) is not satisfied if and only if

$$\begin{cases} (u_1,\dots,u_m) \in \mathtt{U}(m,n) \quad , \qquad \text{Span}\,(u_1,\dots,u_m) = \Gamma^m \\ (a_{21},\dots,a_{m1}) \in \overline{B}^{m-1}(0, \mathscr{K}(r,n,m)) \quad , \qquad v := u_1 + a_{21}u_2 + \dots + a_{m1}u_m \\ h_0^1[v] = h_0^1[u_2] = \dots = h_0^1[u_m] \\ h_0^2[v,v] = h_0^2[v,u_2] = \dots = h_0^2[v,u_m] = 0 \end{cases} \qquad (10.1.3)$$

has no solution.

By construction, the hessian of the restriction $h|_{\Gamma^m}$ has no null eigenvalues, so that system (10.1.3) admits no solution and $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \notin \widehat{\Sigma}^1(r,1,m)$ as wished. This concludes the proof.

10.2 Proof of Theorem C2

10.2.1 Case of a subspace belonging to $\Lambda_1(h,m,n)$

With the usual setting, let $m \in \{2,\dots,n-1\}$ be an integer, and $\Gamma^m \in \Lambda_1(h,m,n) \subset \mathtt{G}(m,n)$ be a m-dimensional subspace spanned by vectors $u_1,\dots,u_m \in \mathtt{U}(m,n)$.

As the Hessian matrix of the restriction $h|_{\Gamma^m}$ has at most one null eigenvalue, without any loss of generality one can choose u_1 to be the eigenvector associated to the unique null eigenvalue, that is

$$\begin{cases} h_0^1[u_1] = h_0^1[u_2] = \cdots = h_0^1[u_m] = 0 \\ h_0^2[u_1,u_1] = h_0^2[u_1,u_2] = h_0^2[u_1,u_m] = 0 \\ \mathrm{Span}(u_1,u_2,\dots,u_m) = \Gamma^m \end{cases} \tag{10.2.1}$$

so that at the same time one must have

$$\det \begin{pmatrix} h_0^2[u_2,u_2] & h_0^2[u_2,u_3] & \dots & h_0^2[u_2,u_m] \\ h_0^2[u_3,u_2] & h_0^2[u_3,u_3] & \dots & h_0^2[u_3,u_m] \\ \dots & & & \\ h_0^2[u_m,u_2] & h_0^2[u_m,u_3] & \dots & h_0^2[u_m,u_m] \end{pmatrix} \neq 0\,. \tag{10.2.2}$$

The expression of $\mathtt{T}_0(h|_{\Gamma^m},r,m)$ w.r.t. the coordinates $x_1,\dots,x_m$ associated to the vectors $u_1,u_2,\dots,u_m$ reads

$$\mathtt{T}_0(h|_{\Gamma^m},r,m)(x) = \sum_{\substack{\mu\in\mathbb{N}^m \\ 1\le|\mu|\le r}} \frac{1}{\mu!} h_0^{|\mu|}\Big[\overbrace{u_1}^{\mu_1},\overbrace{u_2}^{\mu_2},\dots,\overbrace{u_m}^{\mu_m}\Big]\, x_1^{\mu_1}\dots x_m^{\mu_m}\,. \tag{10.2.3}$$

We now claim that

Lemma 10.2.1. *If h is non-steep on Γ^m at some given order $s_m \ge 2$, then*

$$\mathtt{T}_0(h|_{\Gamma^m},r,m) \in X_1^1(r,s_m,m) := \widehat{\sigma}^1(r,s_m,m)\bigcap \mathsf{S}_1^1(r,m) \tag{10.2.4}$$

and $\mathtt{T}_0(h|_{\Gamma^m},r,m)$ satisfies the s_m-vanishing condition on Γ^m on some curve $\gamma \in \widehat{\Theta}_m^1$ whose Taylor expansion at the origin has null linear terms.

Proof. Looking at (10.2.3), it is easy to check that the coefficients of $\mathtt{T}_0(h|_{\Gamma^m},r,m)$ associated to the family of indices $\{\varpi(b,c)\}_{b,c\in\{2,\dots,m\},b\le c}$ introduced in (9.2.1) read

$$p_{\varpi(b,c)} = \big(\mathtt{T}_0(h|_{\Gamma^m},r,m)\big)_{\varpi(b,c)} = \frac{1}{1+\delta_{bc}} h_0^2[u_b,u_c]\,, \tag{10.2.5}$$

where δ_{bc} is the Kronecker delta.

By putting together expressions (10.2.2) - (10.2.5) with the definition of set $\mathsf{S}_1^1(r,m)$ in (9.2.7), one has that

$$\mathtt{T}_0(h|_{\Gamma^m},r,m) \in \mathsf{S}_1^1(r,m)\,. \tag{10.2.6}$$

Since we have assumed h is non-steep on Γ^m at some order $s_m \ge 2$ then, by the discussions at section 7 and at paragraph 9.1, one has

$$\mathtt{T}_0(h|_{\Gamma^m},r,m) \in \widehat{\Sigma}(r,s_m,m) \tag{10.2.7}$$

which, together with (10.2.6), yields

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}(r, s_m, m) \bigcap \mathsf{S}_1^1(r, m) . \tag{10.2.8}$$

We now claim that

Lemma 10.2.2.

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \notin \widehat{\Sigma}^i(r, s_m, m) \qquad \forall\, i \in \{2, \dots, m\} . \tag{10.2.9}$$

Proof. Suppose, by absurd, that $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^i(r, s_m, m)$ for some $i \in \{2, \dots, m\}$. It is clear from Lemma 6.2.3 and (9.1.9) that, as $s_m \geq 2$, one has $\widehat{\sigma}^i(r, s_m, m) \subset \widehat{\sigma}^i(r, 1, m)$, and therefore $\widehat{\Sigma}^i(r, s_m, m) \subset \widehat{\Sigma}^i(r, 1, m)$. This fact and the initial hypothesis imply $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^i(r, 1, m)$, so that by Theorem 9.3.1, one must have

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\sigma}^i(r, 1, m) . \tag{10.2.10}$$

Relation (10.2.10) implies that there must exist a curve $\gamma \in \widehat{\Theta}^i_m$ with values in Γ^m such that $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$ satisfies the 1-vanishing condition on γ. As it was shown in the proof of Theorem C1 (see the discussion around formula (10.1.2)) this is equivalent to asking that system

$$\begin{cases} h_0^1[u_1] = h_0^1[u_2] = \cdots = h_0^1[u_i] = \cdots = h_0^1[u_m] = 0 \\ h_0^2[u_i, u_1] = h_0^1[u_i, u_2] = \cdots = h_0^1[u_i, u_i] = \cdots = h_0^1[u_i, u_m] = 0 \end{cases} \qquad i \neq 1 \tag{10.2.11}$$

admits a solution, which contradicts (10.2.2). ☐

Due to (10.2.8) and to Lemma 10.2.2, we then have that

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^1(r, s_m, m) \bigcap \mathsf{S}_1^1(r, m) . \tag{10.2.12}$$

Moreover, by Theorem 9.3.2, the set $X_1^1(r, s, m) := \widehat{\sigma}^1(r, s, m) \bigcap \mathsf{S}_1^1(r, m)$ defined in (9.3.23) is closed in $\mathsf{S}_1^1(r, m)$ for the induced topology, whence one deduces that actually

$$X_1^1(r, s_m, m) = \text{closure}\left(\widehat{\sigma}^1(r, s_m, m)\right) \bigcap \mathsf{S}_1^1(r, m) = \widehat{\Sigma}^1(r, s_m, m) \bigcap \mathsf{S}_1^1(r, m) . \tag{10.2.13}$$

Relations (10.2.12) and (10.2.13) together imply

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in X_1^1(r, s_m, m) . \tag{10.2.14}$$

Therefore, by (10.2.14) and by the definition of $X_1^1(r, s_m, m)$ in (9.3.23), there exists a curve $\gamma \in \widehat{\Theta}^1_m$, with image in Γ^m, on which the Taylor polynomial of the restriction $\mathtt{T}_0(h|_{\Gamma^m}, r, m)$ satisfies the s_m-vanishing condition. Namely, if $\mathtt{a} = (a_{21}, \dots, a_{m1}) \in$

$\overline{B}^{m-1}(0,\mathscr{K}(r,n,m))$ indicates the linear coefficients of γ and $\widehat{\mathtt{J}}_{s_m,\gamma,\mathtt{a}}$ its s_m-truncation written in the adapted coordinates, by (9.1.9) one must have

$$(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m},r,m),\mathtt{a},\widehat{\mathtt{J}}_{s_m,\gamma,\mathtt{a}}) \in \Upsilon^1(\widehat{Z}^1(r,s_m,m))\,, \tag{10.2.15}$$

that is, by (6.2.33),

$$\widehat{\mathtt{Q}}_{\ell\alpha}(\mathtt{T}_{0,\mathtt{a}}(h|_{\Gamma^m},r,m),\mathtt{a},\widehat{\mathtt{J}}_{s_m,\gamma,\mathtt{a}}) = 0\,,\quad \ell=1,\dots,m\,,\quad \alpha=0,\dots,s_m\,. \tag{10.2.16}$$

In particular, as $s_m \geq 2$, due to Lemma 6.2.3 and to (8.1.5), the equations in (10.2.16) for $\alpha = 0,1$ read

$$\begin{cases} h_0^1[v] = h_0^1[u_1] = \dots = h_0^1[u_m] = 0 \\ h_0^2[v,v] = h_0^2[v,u_2] = h_0^2[v,u_m] = 0 \\ \mathrm{Span}(v,u_2,\dots,u_m) = \Gamma^m \end{cases} \tag{10.2.17}$$

where

$$v = u_1 + \sum_{i=2}^m a_{i1}u_i \tag{10.2.18}$$

is the anisotropic vector associated to the curve γ. Comparing (10.2.1) to (10.2.17), as the Hessian of $h|_{\Gamma^m}$ has only one null eigenvalue we see that the vectors u_1 and v must be parallel, but then (10.2.18) yields

$$v = u_1 \quad , \qquad a_{21} = \dots = a_{m1} = 0\,, \tag{10.2.19}$$

so that by the arguments of subsection (6.2.3) the coordinates adapted to the curve γ coincide with the original coordinates.

□

We now recall that, due to Theorem 9.3.2, the form of the set

$$Y_1^1(r,s_m,m) := \Pi_{\widehat{W}^1(r,m)}\left(\mathscr{N}\left(\{\widehat{\mathtt{Q}}_{\mathtt{i}\alpha}\}_{\substack{\mathtt{i}\in\{1,\dots,m\}\\ \alpha\in\{0,\dots,s_m\}}}\right)\bigcap \Upsilon^1(\mathtt{S}_1^1(r,m)\times\widehat{\vartheta}_m^1(s))\right) \tag{10.2.20}$$

satisfying[1]

$$X_1^1(r,s_m,m) = \Pi_{\mathcal{P}(r,m)}\left((\mathfrak{U}^1)^{-1}Y_1^1(r,s_m,m)\right) \tag{10.2.21}$$

can be explicitly computed - starting from the relations determining $\Upsilon^1(\widehat{Z}^1(r,s_m,m))$ - by the means of an algorithm involving only linear operations. By this fact, the form of the set

$$\mathscr{Y}_1^1(r,s_m,m) := Y_1^1(r,s_m,m)\bigcap\{(\mathtt{p}_\mu,\mathtt{a})\in\widehat{W}^1(r,m)|\mathtt{a}=0\} \tag{10.2.22}$$

[1]The transformation $\mathfrak{U}^1$ was introduced in Remark 6.2.3.

can also be deduced explicitly starting from $\Upsilon^1(\widehat{Z}^1(r, s_m, m))$ and, due to Lemma 10.2.1 and to (10.2.21), one has that

$$\begin{array}{c} h \text{ non-steep} \\ \text{at order } s_m \text{ on } \Gamma^m \end{array} \implies \mathfrak{U}^1(\mathtt{T}_0(h|_{\Gamma^m}, r, m), 0) = (\mathtt{T}_0(h|_{\Gamma^m}, r, m), 0) \in \mathscr{Y}_1^1(r, s_m, m) \, . \tag{10.2.23}$$

Moreover, we observe the following facts:

1. the explicit expression of set $\mathcal{Z}_n^{r,s_m,m} \subset \mathcal{P}^\star(r,n) \times \mathbb{R}^{(m-1)s_m} \times \mathscr{V}^1(m,n)$ introduced in Corollary B2 can be obtained by injecting into the explicit expression for

$$\Upsilon^1(\widehat{Z}^1(r, s_m, m)) := \mathscr{N}\left\{ \mathtt{Q}_{i,\alpha}(\mathtt{T}_{0,\mathtt{a}}(\mathtt{P}_\mathtt{a}, r, m), \mathtt{a}, \mathtt{J}_{s_m,\gamma,\mathtt{a}}) \right\}_{\substack{i=1,\dots,m \\ \alpha=0,\dots,s_m}} \tag{10.2.24}$$

given in Lemma 6.2.3 the explicit form of the coefficients of $\mathtt{T}_{0,\mathtt{a}}(P|_{\Gamma^m}, r, m)$ in (10.1.1), with P any polynomial belonging to $\mathcal{P}^\star(r,n)$, and by considering the vectors $v, u_2, \dots, u_m$ in (10.1.1) as free parameters of $\mathscr{V}^1(m,n)$.

2. In the same way, for any $P \in \mathcal{P}^\star(r,n)$, one can inject in the expressions determining $\mathscr{Y}_1^1(r, s_m, m)$ the explicit form of the coefficients of $\mathtt{T}_{0,\mathtt{a}}(P|_{\Gamma^m}, r, m)$, given in (10.1.1), with the vectors $v, u_2, \dots, u_m$ considered as free parameters of $\mathscr{V}^1(m,n)$. The resulting subset is indicated by

$$\mathscr{A}_1(r, s_m, n, m) \subset \mathcal{P}^\star(r,n) \times \mathbb{R}^{m-1} \times \mathscr{V}^1(m,n) \, .$$

By the arguments above, and by the fact that the form of $\mathscr{Y}_1^1(r, s_m, m)$ is obtained starting from the expression of $\Upsilon^1(\widehat{Z}^1(r, s_m, m))$ through linear algorithms, we have that the explicit expression of $\mathscr{A}_1(r, s_m, n, m)$ can be found linearly starting from the expressions determining $\mathcal{Z}_n^{r,s_m,m}$.

Furthermore, the above arguments together with (10.2.23) and with formula (10.2.19) yield that if system

$$\begin{cases} (u_1, \dots, u_m) \in \mathbb{U}(m,n) \quad , \qquad \text{Span}\,(u_1, u_2, \dots, u_m) = \Gamma^m \in \Lambda_1(h, m, n) \\ (\mathtt{T}_0(h, r, n), 0, u_1, u_2, \dots, u_m) \in \mathscr{A}_1(r, s_m, n, m) \end{cases} \tag{10.2.25}$$

has no solution, then h is steep around the origin with index $\alpha_m \le 2s_m - 1$ on any subspace $\Gamma^m \in \Lambda_1(h, m, n)$.

10.2.2 Case of a subspace belonging to $\Lambda_2(h, m, n)$

As we did in paragraph 10.2.1, we choose a basis $u_1, \dots, u_m \in \mathbb{U}(m,n)$ spanning Γ^m such that u_1 coincides with a normalized eigenvector of the hessian associated to one of the null eigenvalues. Then, we have

Lemma 10.2.3. *If h is non-steep on $\Gamma^m \in \Lambda_2(h,m,n)$ at a given order $s_m \geq 2$, up to suitably changing the order of the vectors $u_1, \dots, u_m$ one has*

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^1(r, s_m, m) \bigcap \mathtt{S}_2^1(r,m) \,. \tag{10.2.26}$$

Proof. By (10.2.5), and by the fact that the hessian of $h|_{\Gamma^m}$ has two or more null eigenvalues, we have that

$$\det \begin{pmatrix} h_0^2[u_2,u_2] & h_0^2[u_2,u_3] & \dots & h_0^2[u_2,u_m] \\ h_0^2[u_3,u_2] & h_0^2[u_3,u_3] & \dots & h_0^2[u_3,u_m] \\ \dots & & & \\ h_0^2[u_m,u_2] & h_0^2[u_m,u_3] & \dots & h_0^2[u_m,u_m] \end{pmatrix} = 0 \,, \tag{10.2.27}$$

hence $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \mathtt{S}_2^1(r,m)$ following definition (9.2.9).

Moreover, h is non-steep on Γ^m at a given order $s_m \geq 2$ so that, by the discussions at section 7 and at paragraph 9.1, one has

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}(r, s_m, m) \tag{10.2.28}$$

so that by the previous considerations we have

$$\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}(r, s_m, m) \bigcap \mathtt{S}_2^1(r,m) \,. \tag{10.2.29}$$

By the above expression and by (9.1.9), we have that there must exist $i \in \{1, \dots, m\}$ such that $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^i(r, s_m, m) \bigcap \mathtt{S}_2^1(r,m)$. As it was already discussed in the proof of Theorem C1, if $\mathtt{T}_0(h|_{\Gamma^m}, r, m) \in \widehat{\Sigma}^i(r, s_m, m) \bigcap \mathtt{S}_2^1(r,m)$ then the vector u_i must satisfy (10.2.11). If $i=1$, there is nothing else to prove. If $i \neq 1$, it suffices to change the order of the vectors u_1 and u_i. □

Now, if for any $P \in \mathcal{P}^\star(r,n)$ we inject in the expressions[2] determining the semi-algebraic subset $\mathfrak{U}^1\left(\left(\widehat{\Sigma}^1(r,s_m,m) \bigcap \mathtt{S}_2^1(r,m)\right) \times \widehat{\vartheta}_m^1(1)\right) \subset \widehat{W}^1(r,m)$ the explicit expression of the coefficients of the polynomial $\mathtt{T}_{0,\mathtt{a}}(P|_{\Gamma^m}, r, m)$, and we let the vectors $v, u_2, \dots, u_m$ appearing in (10.1.1) vary in $\mathscr{V}^1(m,n)$, we obtain a set

$$\mathscr{A}_2(r, s_m, n, m) \subset \mathcal{P}(r,n) \times \mathbb{R}^{m-1} \times \mathscr{V}^1(m,n) \,.$$

Moreover, by Lemma 10.2.3, we have that if system

$$\begin{cases} (a_{21}, \dots, a_{m1}) \in \overline{B}^{m-1}(\mathscr{K}) \\ (u_1, \dots, u_m) \in \mathtt{U}(m,n) \quad , \qquad v := u_1 + \sum_{i=2}^m a_{i1} u_i \\ \text{Span}\,(v, u_2, \dots, u_m) = \Gamma^m \in \Lambda_2(h,m,n) \\ (\mathtt{T}_0(h,r,n), a_{21}, \dots, a_{m1}, v, u_2, \dots, u_m) \in \mathscr{A}_2(r, s_m, n, m) \end{cases} \tag{10.2.30}$$

has no solution, then h is steep around the origin with index $\alpha_m \leq 2s_m - 1$ on any subspace $\Gamma^m \in \Lambda_2(h,m,n)$.

[2] Contrary to the case studied in the previous paragraph, here these expressions cannot be found easily, in general.

10.3 Proof of Theorem C3

10.3.1 Construction of an atlas of analytic maps for the Grassmannian

It is well known that for any choice of positive integers k, n, with $k < n$, the Grassmannian $\mathtt{G}(k, n)$ has the structure of a projective algebraic variety (see e.g. [27], [100]). In this subparagraph, for any positive integer $n \geq 3$ and for any $m \in \{2, \dots, n-1\}$ we will construct a suitable atlas of analytic maps for $\mathtt{G}(m, n)$ by exploiting classic arguments of real-algebraic geometry.

Namely, we fix two integers $n \geq 3$, and $m \in \{2, \dots, n-1\}$ and we consider a subset $I := (i_1, \dots, i_m) \subset \{1, \dots, n\}$ of cardinality m, as well as its complementary $J := (j_1, \dots, j_{n-m})$ of cardinality $n - m$ in $\{1, \dots, n\}$.

We work in the euclidean space $\mathbb{R}^n$, and we fix a bilinear, symmetric, non-degenerate form $\mathsf{B} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$. The Spectral Theorem ensures the existence of an orthonormal basis - indicated by $e_1, \dots, e_n$ - that diagonalizes B. Namely, in the basis $e_1, \dots, e_n$ the form B is represented by matrix

$$\mathsf{B}_{(e_1,\dots,e_n)} := \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_n \end{pmatrix}, \tag{10.3.1}$$

where $\alpha_1 \times \dots \times \alpha_n \neq 0$.

We indicate by E_I (resp. E_J) the m-dimensional subspace spanned by the vectors $(e_{i_1}, \dots, e_{i_m})$ (resp. the $n - m$-dimensional subspace spanned by $e_{j_1}, \dots, e_{j_{n-m}}$). One clearly has $\mathbb{R}^n = E_I \oplus E_J$. We also denote by U_J the subset of $\mathtt{G}(m, n)$ containing the m-dimensional subspaces which are supplementary for E_J.

With this setting, we consider the cartesian product $E_J^m := \overbrace{E_J \times \dots \times E_J}^{m \text{ times}}$ and we have that

Lemma 10.3.1. *The map*

$$\mathscr{F}_J : E_J^m \longrightarrow U_J \qquad (w_1, \dots, w_m) \longmapsto Span(e_{i_1} + w_1, \dots, e_{i_m} + w_m) \tag{10.3.2}$$

is bijective.

Proof. We proceed by steps. In the first two steps, we check that definition (10.3.2) is well-posed. In Steps 3 and 4 we prove injectivity and surjectivity.

Step 1. We claim that for any choice of $(w_1, \dots, w_m) \in E_J^m$, the vectors $(e_{i_1} + w_1, \dots, e_{i_m} + w_m)$ are linearly independent. Infact, if by absurd there exist m vectors $(w_1, \dots, w_m) \in E_J^m$ such that $(e_{i_1} + w_1, \dots, e_{i_m} + w_m)$ are not linearly independent, then there must be a vector $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \backslash \{0\}$ satisfying $\sum_{\ell=1}^m \lambda_\ell (e_{i_\ell} + w_\ell) = 0$, that is $\sum_{\ell=1}^m \lambda_\ell e_{i_\ell} = -\sum_{\ell=1}^m \lambda_\ell w_{i_\ell}$.

As $\sum_{\ell=1}^m \lambda_\ell e_{i_\ell} \in E_I$, and $-\sum_{\ell=1}^m \lambda_\ell w_{i_\ell} \in E_J$, and as $\mathbb{R}^n = E_I \oplus E_J$ by construction, by the assumptions one must have $-\sum_{\ell=1}^m \lambda_\ell w_{i_\ell} = \sum_{\ell=1}^m \lambda_\ell e_{i_\ell} = 0$. The previous relation - together with the fact that the vectors $e_{i_1}, \dots, e_{i_m}$ are linearly independent by hypothesis, yields $\lambda = 0$, in contradiction with the initial assumption $\lambda \neq 0$.

Step 2. We claim that for any choice of $(w_1, \dots, w_m) \in E_J^m$ one has $\mathrm{Span}\,(e_{i_1} + w_1, \dots, e_{i_m} + w_m) \in U_J$.

By absurd, we suppose that for some $(w_1, \dots, w_m) \in E_J^m$ there exist two non-zero vectors $u \in E_J$, $\sigma = (\sigma_1, \dots, \sigma_m) \in \mathbb{R}^m$, verifying $u = \sum_{\ell=1}^m \sigma_\ell (e_{i_\ell} + w_\ell)$, that is $\sum_{\ell=1}^m \sigma_\ell e_{i_\ell} = u - \sum_{\ell=1}^m \sigma_\ell w_\ell$. By construction, one has $\sum_{\ell=1}^m \sigma_\ell e_{i_\ell} \in E_I$, and $u - \sum_{\ell=1}^m \sigma_\ell w_\ell \in E_J$. Hence, as $\mathbb{R}^n = E_I \oplus E_J$, the previous formula yields $u - \sum_{\ell=1}^m \sigma_\ell w_\ell = \sum_{\ell=1}^m \sigma_\ell e_{i_\ell} = 0$, which in turn implies $\sigma = 0$, as the vectors $(e_{i_1}, \dots, e_{i_m})$ are linearly independent.

Hence, one has $E_J \cap \mathrm{Span}\,(e_{i_1} + w_1, \dots, e_{i_m} + w_m) = \{0\}$. Therefore, since $\dim E_J = n - m$ and $\dim \Big(\mathrm{Span}\,(e_{i_1} + w_1, \dots, e_{i_m} + w_m) \Big) = m$ (the vectors $e_{i_1} + w_1, \dots, e_{i_m} + w_m$ are linearly independent by Step 1), the subspace given by $\mathrm{Span}\,(e_{i_1} + w_1, \dots, e_{i_m} + w_m)$ is supplementary to E_J and thus belongs to U_J.

Step 3. We prove that $\mathscr{F}_J$ is injective. By absurd, we suppose that there exists a subspace in U_J which has two different pre-images. Namely, we suppose that there exist vectors $(u_1, \dots, u_m) \in E_J^m$, and $(w_1, \dots, w_m) \in E_J^m$, with $(u_1, \dots, u_m) \neq (w_1, \dots, w_m)$, satisfying $\mathscr{F}_J(u_1, \dots, u_m) = \mathscr{F}_J(w_1, \dots, w_m)$. Hence, as $e_{i_1} + u_1, \dots, e_{i_m} + u_m$ and $e_{i_1} + w_1, \dots, e_{i_m} + w_m$ generate the same subspace, for any $\ell \in \{1, \dots, m\}$ there must exist $\beta^\ell = (\beta_1^\ell, \dots, \beta_m^\ell) \in \mathbb{R}^m \backslash \{0\}$ such that

$$e_{i_\ell} + w_\ell = \sum_{k=1}^m \beta_k^\ell (e_{i_k} + u_k) \,, \tag{10.3.3}$$

that is

$$e_{i_\ell} - \sum_{k=1}^m \beta_k^\ell e_{i_k} = \sum_{i=1}^m \beta_i^\ell u_i - w_\ell \,. \tag{10.3.4}$$

By construction one has $e_{i_\ell} - \sum_{k=1}^m \beta_k^\ell e_{i_k} \in E_I$ and $\sum_{i=1}^m \beta_i^\ell u_i - w_\ell \in E_J$, so due to (10.3.4) and to the direct sum $\mathbb{R}^n = E_I \oplus E_J$ one infers $\sum_{i=1}^m \beta_i^\ell u_i - w_\ell = e_{i_\ell} - \sum_{k=1}^m \beta_k^\ell e_{i_k} = 0$. Since the vectors $e_{i_1}, \dots, e_{i_m}$ are linearly independent, we finally obtain

$$\beta_k^\ell = \delta_{\ell k} \,. \tag{10.3.5}$$

where $\delta_{\ell k}$ is the Kronecker symbol. Formulas (10.3.3) and (10.3.5) together imply that

$$e_{i_\ell} + w_\ell = e_{i\ell} + u_\ell \quad \Longleftrightarrow \quad w_\ell = u_\ell \qquad \forall \ell \in \{1, \dots, m\} \,, \tag{10.3.6}$$

in contradiction with the hypothesis $(u_1, \dots, u_m) \neq (w_1, \dots, w_m)$.

Step 4. We prove that $\mathscr{F}_J$ is surjective. Consider a subspace $V \in U_J$. Since V is supplementary of E_J, one has the direct sum $\mathbb{R}^n = V \oplus E_J$, and for any $\ell \in \{1, \dots, m\}$ there exist unique vectors $(v_\ell, w_\ell) \in V \times E_J$ such that $e_{i_\ell} = v_\ell - w_\ell$. Hence, for any $\ell \in \{1, \dots, m\}$ there exists a unique choice of vectors $(w_1, \dots, w_m)$ satisfying $v_\ell = e_{i_\ell} + w_\ell$. The vectors $v_1, \dots, v_m$ belong to V by construction, and are linearly independent by Step 1. Therefore, since $\dim V = m$ by hypothesis ($V \in U_J$), one has $\mathrm{Span}(v_1, \dots, v_m) = V$. □

We indicate by $\mathscr{J}^{n-m}$ the subsets of cardinality $n-m$ in $\{1, \dots, n\}$. One has the following covering of the m-dimensional Grassmannian:

$$\mathtt{G}(m,n) = \bigcup_{J \in \mathscr{J}^{n-m}} U_J \,. \tag{10.3.7}$$

By construction, any vector $w_\ell \in E_J$ can be expressed uniquely as

$$w_\ell = \sum_{k=1}^{n-m} M_{\ell k} e_{j_k} \,, \tag{10.3.8}$$

where $(M_{\ell k})_{\substack{\ell=1,\dots,m \\ k=1,\dots,n-m}}$ is a real $m \times (n-m)$ matrix. By (10.3.7), and by Lemma 10.3.1, there exists an atlas sending $\mathtt{G}(m,n)$ to the open union

$$\bigcup_{J \in \mathscr{J}^{n-m}} \mathscr{F}_J^{-1}(U_J) = \bigcup_{J \in \mathscr{J}^{n-m}} E_J^m \subset \mathbb{R}^{m \times (n-m)} \,. \tag{10.3.9}$$

10.3.2 Proof of Theorem C3

Taking (10.3.7) into account, we fix $J \in \mathscr{J}^{n-m}$ together with its associated sets E_J, U_J. Let V be a m-dimensional subspace belonging to U_J. By Lemma 10.3.1, one has

$$V = \mathrm{Span}\,\{e_{i_1} + w_1, \dots, e_{i_m} + w_m\} \tag{10.3.10}$$

for a unique $(w_1, \dots, w_m) \in E_J^m$.

Now, as $\mathtt{G}_1(m,n)$ contains those subsets of $\mathtt{G}(m,n)$ on which the restriction of the bilinear form B has at least one null eigenvalue, $V \in \mathtt{G}_1(m,n)$ if and only if B is degenerated on V. Namely, taking (10.3.10) into account, $V \in \mathtt{G}_1(m,n)$ iff there exists a vector $v = \sum_{\ell=1}^m v_\ell (e_{i_\ell} + w_\ell) \in V$ such that for all $\ell' \in \{1, \dots, m\}$ one has

$$\begin{aligned} \mathsf{B}(v, e_{i_{\ell'}} + w_{\ell'}) &= \sum_{\ell=1}^m v_\ell\, \mathsf{B}(e_{i_\ell} + w_\ell, e_{i_{\ell'}} + w_{\ell'}) \\ &= \sum_{\ell=1}^m v_\ell \left(\mathsf{B}(e_{i_\ell}, e_{i_{\ell'}}) + \mathsf{B}(w_\ell, w_{\ell'}) \right) \\ &= v_{\ell'}\, \alpha_{i_{\ell'}} + \sum_{\ell=1}^m v_\ell\, \mathsf{B}(w_\ell, w_{\ell'}) \,. \end{aligned} \tag{10.3.11}$$

To pass from the first to the second line in the above expression, we have taken into account the fact that $(w_1, \dots, w_m) \in E_J^m$, that $E_J = \mathrm{Span}(e_{j_1}, \dots, e_{j_{n-m}})$, and that the form B is diagonal in the basis $e_1, \dots, e_m$; in the last passage, we have considered (10.3.1). Setting

$$\mathscr{M}_{\mathsf{B}} = \mathscr{M}_{\mathsf{B}}(w_1, \dots, w_m) := \begin{pmatrix} \frac{1}{\alpha_{i_1}}\mathsf{B}(w_1, w_1) & \dots & \frac{1}{\alpha_{i_1}}\mathsf{B}(w_1, w_m) \\ \dots & \dots & \dots \\ \frac{1}{\alpha_{i_m}}\mathsf{B}(w_m, w_1) & \dots & \frac{1}{\alpha_{i_m}}\mathsf{B}(w_m, w_m) \end{pmatrix} \tag{10.3.12}$$

it is plain to check that (10.3.11) can be rewritten in the form

$$\mathscr{M}_{\mathsf{B}}\, v = -v \tag{10.3.13}$$

that is, we are asking for -1 to be an eigenvalue of $\mathscr{M}_{\mathsf{B}}$, hence (10.3.13) is equivalent to

$$\det(\mathscr{M}_{\mathsf{B}} + \mathbb{I}_m) = 0\,, \tag{10.3.14}$$

where $\mathbb{I}_m$ is the $m \times m$ identity matrix.

Since $\mathscr{M}_{\mathsf{B}}$ depends on $(w_1, \dots, w_m) \in E_J^m$ and since $(w_1, \dots, w_m)$ are in bijection with $\mathbb{R}^{m(n-m)}$ by (10.3.8), the quantity $\det(\mathscr{M}_{\mathsf{B}} + \mathbb{I}_m)$ determines a polynomial map $\mathbb{R}^{m(n-m)} \longrightarrow \mathbb{R}$.

If, by absurd, $\det(\mathscr{M}_{\mathsf{B}} + \mathbb{I}_m)$ is the null polynomial, then relation (10.3.14) holds on the whole inverse image $\mathscr{F}_J^{-1}(U_J)$. In particular, we observe that $(w_1 = 0, \dots, w_m = 0) \in \mathscr{F}_J^{-1}(U_J)$ because $\mathscr{F}_J(0, \dots, 0) = \mathrm{Span}\{e_{i_1}, \dots, e_{i_m}\} = E_I$ and E_I is supplementary of E_J by construction. Therefore, by the above reasonings one must have

$$\det(\mathscr{M}_{\mathsf{B}}(0, \dots, 0) + \mathbb{I}_m) = \det \mathbb{I}_m = 0$$

which is clearly false. Consequently, the polynomial function $\det(\mathscr{M}_{\mathsf{B}}(w_1, \dots, w_m) + \mathbb{I}_m)$ is not identically null over $\mathbb{R}^{m(n-m)}$ and, due to Lemma C.2.3, its zero set is contained in a submanifold of codimension one in $\mathbb{R}^{m(n-m)}$. Hence, also the subset of degenerated subspaces of U_J is contained in a submanifold of codimension one in $\mathtt{G}(m, n)$. The reasoning can be repeated for all $J \in \mathscr{J}^{n-m}$. As, by its definition and by (10.3.7), $\mathtt{G}_1(m, n)$ is the finite union over $J \in \mathscr{J}^{n-m}$ of the degenerated subspaces of U_J, we have that $\mathtt{G}_1(m, n)$ is contained in a submanifold of codimension one in $\mathtt{G}(m, n)$. This proves point 1 of the statement.

With the setting above, for any fixed $J \in \mathscr{J}^{n-m}$ we observe that a m-dimensional subspace $V' \in U_J$ belongs to $\mathtt{G}_2(m, n)$ if and only if there exist at least two linearly independent vectors $v, u \in V'$ satisfying (10.3.13). In particular, the subset $\mathtt{G}_2(m, n) \cap U_J$ of "doubly-degenerated" subspaces of U_J is contained in the intersection of $\mathtt{G}_1(m, n) \cap U_J$ with the set

$$T_J := \{W \in U_J | \Delta(P_{\mathscr{M}_{\mathsf{B}}}) = 0\} \tag{10.3.15}$$

where $P_{\mathscr{M}_\mathsf{B}}$ is the characteristic polynomial of matrix $\mathscr{M}_\mathsf{B}$, and $\Delta(P_{\mathscr{M}_\mathsf{B}})$ is its discriminant. By the above arguments and by (10.3.7), point 2 of the statement follows if we manage to prove that T_J is contained in a submanifold of codimension one in $\mathtt{G}(m, n)$. The rest of the proof will be devoted to demonstrating this property.

Clearly, by the same arguments used in the proof of point 1 of the statement, $\Delta(P_{\mathscr{M}_\mathsf{B}})$ is a polynomial function over $\mathbb{R}^{m(n-m)}$. If, by absurd, $\Delta(P_{\mathscr{M}_\mathsf{B}})$ is identically zero in $\mathbb{R}^{m(n-m)}$, then in particular it must be zero over the open set $\mathscr{F}_J^{-1}(U_J)$.

Now, choose m numbers $j_1, \dots, j_m \in J$, and consider the vectors

$$w'_1 := \sqrt{\left|\frac{\alpha_{i_1}}{\alpha_{j_1}}\right|}e_{j_1}, \quad w'_2 := \sqrt{2\left|\frac{\alpha_{i_2}}{\alpha_{j_2}}\right|}e_{j_2}, \quad \dots, \quad w'_m := \sqrt{m\left|\frac{\alpha_{i_m}}{\alpha_{j_m}}\right|}e_{j_m}\,, \tag{10.3.16}$$

which are well defined by the fact that B is non-degenerate (see (10.3.1)). It is clear that $(w'_1, \dots, w'_m) \in E_J^m$, so that by Lemma (10.3.1) one has $\mathscr{F}_J(w'_1, \dots, w'_m) \in U_J$. As B is diagonal for the basis $e_1, \dots, e_n$ by hypothesis, matrix $\mathscr{M}_\mathsf{B}(w'_1, \dots, w'_m)$ in (10.3.1) reads

$$\begin{pmatrix} \operatorname{sgn}(\alpha_{i_1})\operatorname{sgn}(\alpha_{j_1}) & 0 & 0 & \dots & 0 \\ 0 & 2\operatorname{sgn}(\alpha_{i_2})\operatorname{sgn}(\alpha_{j_2}) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & m\operatorname{sgn}(\alpha_{i_m})\operatorname{sgn}(\alpha_{j_m}) \end{pmatrix}, \tag{10.3.17}$$

and it is clear that the discriminant of the characteristic polynomial of this matrix cannot be zero, in contradiction with the fact that $\Delta(P_{\mathscr{M}_\mathsf{B}}) = 0$ on the whole set $\mathscr{F}_J^{-1}(U_J)$. Therefore, the polynomial function $\Delta(P_{\mathscr{M}_\mathsf{B}})$ is not identically null over $\mathbb{R}^{m(n-m)}$ and its zero set is contained in a submanifold of codimension one in $\mathbb{R}^{m(n-m)}$, by Lemma C.2.3. This proves that T_J is contained in a submanifold of codimension one in $\mathtt{G}(m, n)$, which concludes the proof.

Part II

Bernstein-Remez inequality for algebraic functions: a topological approach.

Abstract

By taking full advantage of the structure of complex algebraic curves and by using compactness arguments, in this part we give a self-contained proof that holomorphic algebraic functions verify a uniform Bernstein-Remez inequality. Namely, their growth over a bounded, open, complex set is uniformly controlled by their size on a compact complex subset of sufficiently high cardinality. Up to our knowledge, the first known demonstration on the existence of such an inequality for a specific subset of algebraic functions is contained in Nekhoroshev's 1973 breakthrough on the genericity of close-to-integrable Hamiltonian systems that are stable over long time. Despite its pivotal rôle, this passage of Nekhoroshev's proof has remained unnoticed so far. This work aims at extending and generalizing Nekhoroshev's arguments to a modern framework. We stress the fact that our proof is different from the one contained in Roytwarf and Yomdin's seminal work (1998), where Bernstein-type inequalities are proved for several classes of functions.

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Notice

This part of the thesis can be found online as a pre-print. Its reference is

S. Barbieri, L. Niederman, *Bernstein-Remez inequality for algebraic functions: a complex analytic approach*

https://hal.science/hal-03739568/file/Bernstein_Remez.pdf

Chapter 11

Introduction and main result

11.1 The Bernstein-Remez inequality

Let $\Omega \subset \mathbb{C}$ be an open bounded domain, $\mathcal{K} \subset \Omega$ be a compact subset and let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic in Ω and continuous in its closure $\overline{\Omega}$. The *Bernstein's constant of f* with respect to $\Omega, \mathcal{K}$ is the quantity

$$\mathtt{B}(f, \mathcal{K}, \Omega) := \max_{\overline{\Omega}} |f| / \max_{\mathcal{K}} |f| \ .$$

Any family $\mathcal{F}$ of holomorphic functions defined in Ω and continuous in $\overline{\Omega}$ is said to satisfy a *uniform Bernstein-Remez inequality* if there exists $\mathtt{C}(\mathcal{K}, \Omega) > 0$ such that for all $f \in \mathcal{F}$

$$\max_{\overline{\Omega}} |f| \leq \mathtt{C}(\mathcal{K}, \Omega) \max_{\mathcal{K}} |f| \qquad \text{or, equivalently, if} \qquad \sup_{f \in \mathcal{F}} \mathtt{B}(f, \mathcal{K}, \Omega) \leq \mathtt{C}(\mathcal{K}, \Omega) \ .$$

The term *Bernstein-Remez inequality* is used in order to avoid confusion with other sorts of Bernstein's inequalities that involve derivatives or primitives (see e.g. [58]).

The Bernstein-Remez inequality and the existence of families verifying a uniform estimate of this kind turn out to be important in many areas of mathematics. Without pretending to make a complete survey on the subject, we observe that these kind of estimates appear in the study of the local behavior of certain holomorphic functions (see e.g. [113], [40], [93], [51], [42], [101], [52]), in questions related to the second part of Hilbert's 16th problem (see e.g. [62], [40], [76], [41], [61]), in the study of special classes of ODEs (see e.g. [77]) and subelliptic PDEs (see e.g. [58], [59]), as well as in potential theory (see e.g. [109], [43]) and in dynamical systems when investigating questions related to entropy (see e.g. [115]).

In this part, we are interested in finding a family of functions verifying a uniform Bernstein-Remez inequality. Namely, by extending a strategy due to Nekhoroshev [94] and that is different from the known demonstrations in this field (see [107], [38], [117], [43]), with the above notations we shall prove the following.

If

1. the graph of f solves the algebraic equation $S(z, f(z)) = 0$ for some non-zero polynomial $S \in \mathbb{C}[X, Y]$ of degree k;

2. the algebraic curve of S over Ω is given by the union of vertical lines of the form $\{(z, w) \in \mathbb{C}^2 \mid z = z_*\}$ together with disjoint graphs of holomorphic functions over Ω;

3. the cardinality of $\mathcal{K}$ is strictly greater than k;

then the Bernstein's constant of f w.r.t. $\Omega, \mathcal{K}$ depends on k but is independent of f.

Before stating this result more rigorously (see Theorem 11.5.1), let us discuss our motivation for developing this subject.

11.2 Rôle in Hamiltonian dynamics and Nekhoroshev theory

The authors discovered the Bernstein-Remez inequality during the investigation of an important result of Hamiltonian dynamics. However, before describing the key rôle played by the Bernstein-Remez estimate in this field, we make a short review of some general results which are helpful in order to make the context clear to the reader.

Namely, Hamiltonian formalism is the natural setting appearing in the study of many physical systems. In the simplest case, we consider the motion of a point on a Riemannian manifold $\mathcal{M}$, called configuration manifold, governed by Newton's second law ($\ddot{q} = -\nabla U(q)$ for a potential function U in the euclidean case, with q a system of local coordinates for $\mathcal{M}$). This system can be transformed by duality thanks to Legendre's transformation and reads

$$\dot{p} = -\partial_q H(p, q) \quad , \qquad \dot{q} = \partial_p H(p, q) \quad ,$$

where $H(p, q)$ is a real differentiable function on the cotangent bundle $T^*\mathcal{M}$, classically called Hamiltonian, and p is the coordinate conjugated to q. Systems integrable by quadrature are an important class of Hamiltonian systems. A Hamiltonian system depending on $2n$ variables (n degrees of freedom) is said to be integrable in the sense of Arnol'd-Liouville if it can be conjugated to a Hamiltonian system on the cotangent bundle of the n-dimensional torus $\mathbb{T}^n$, whose equations of motion take the form

$$\dot{I} = -\partial_\vartheta h(I) = 0 \quad , \qquad \dot{\vartheta} = \partial_I h(I) \quad ,$$

where $(I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ are called action-angle coordinates. Therefore, the phase space for an integrable system is foliated by invariant tori carrying the linear motions of the angular variables (called quasi-periodic motions). Integrable systems are exceptional, but many important physical problems are governed by Hamiltonian systems which are close to integrable. Namely, the dynamics of a nearly-integrable Hamiltonian system is described by a Hamiltonian function whose form in action-angle coordinates $(I, \vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ reads

$$H(I, \vartheta) := h(I) + \varepsilon f(I, \vartheta) \,,$$

where ε is a small parameter. The structure of the phase space for this kind of systems can be inferred with the help of Kolmogorov-Arnol'd-Moser (KAM) theory. Namely, under a generic non-degeneracy condition for h, a Cantor set of large measure of invariant tori carrying quasi-periodic motions for the integrable flow persists under a suitably small perturbation (see e.g. ref. [5], [46]).

For systems with three or more degrees of freedom, KAM theory yields little information about trajectories lying in the complementary of such Cantor set, where instabilities may occur (see e.g. ref. [3]). However, in a series of articles published during the seventies (see ref. [95], [96], or [70], [14] for a more modern presentation), Nekhoroshev proved an effective result of stability for all initial conditions holding over a time which is exponentially long in the inverse of the size ε of the perturbation, provided that the Hamiltonian is analytic and that its integrable part satisfies a generic transversality property known as *steepness*.

In order to introduce the steepness property, we fix a positive integer $n \geq 2$ and we indicate by $B^n(0, R)$ the real n-dimensional ball of radius R centered at the origin. Then, we have

Definition 11.2.1 (Steepness)**.** Fix $\delta > 0$, $R > 0$. A C^2 function $h : B^n(0, R{+}2\delta) \to \mathbb{R}$ is steep in $B^n(0, R)$ with steepness indices $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1} \geq 1$ and *steepness coefficients* $C_1, \dots, C_{n-1}, \delta$ if:

1. $\inf_{I \in B^n(0,R)} ||\nabla h(I)|| > 0$;
2. for any $I \in B^n(0, R)$, for any integer $1 \leq m < n$, and for any m-dimensional subspace Γ^m orthogonal to $\nabla h(I)$ and endowed with the induced euclidean metric, one has:

$$\max_{0 \leq \eta \leq \xi} \; \min_{u \in \Gamma^m,\, ||u||_2 = \eta} || \, \pi_{\Gamma^m} \nabla h(I+u) \, || > C_m \xi^{\boldsymbol{\alpha}_m}, \quad \forall \xi \in (0, \delta], \tag{11.2.1}$$

   where π_{Γ^m} stands for the orthogonal projection on Γ^m.

Remark 11.2.1. Since in definition 11.2.1 the subspace $\Gamma^m \subset \mathbb{R}^n$ is endowed with the induced metric, for all $u \in \Gamma^m$ one has $||\pi_{\Gamma^m} \nabla h(I+u)|| = ||\nabla(h|_{I+\Gamma^m})(I+u)||$, where $h|_{I+\Gamma^m}$ indicates the restriction of h to the affine subspace $I + \Gamma^m$.

Remark 11.2.2. It is worth mentioning that a real-analytic function is steep if and only if it has no isolated critical points and if any of its restrictions to any affine proper subspace has only isolated critical points (see [75] and [98]).

With this notion, Nekhoroshev's effective result of stability reads

Theorem 11.2.1 (Nekhoroshev, 1977)**.** *Consider a nearly-integrable system with Hamiltonian $H(I,\vartheta) := h(I)+\varepsilon f(I,\vartheta)$ analytic in some complex neighborhood of $B^n(0,R)\times \mathbb{T}^n$, and assume that h is steep. Then, there exist positive constants $a, b, \varepsilon_0, C_1, C_2, C_3$ such that, for any $\varepsilon \in [0,\varepsilon_0)$ and for any initial condition not too close to the boundary, one has $|I(t) - I(0)| \le C_2\varepsilon^a$ for any time t satisfying $|t| \le C_1 \exp\left(C_3/\varepsilon^b\right)$.*

Nekhoroshev also proved in [94] that the steepness condition is generic, both in measure and in topological sense: for a sufficiently high positive integer r, the Taylor polynomials of order less or equal than r of non-steep functions are contained in a semi-algebraic[1] set having positive codimension in the space of polynomials of order bounded by r. Hence, steep functions are characterised by the fact that their Taylor polynomials satisfy suitable algebraic conditions (see [96] and [12]). Although these results have been studied and extended for more than forty years (so that *Nekhoroshev Theory* is a classic subject of study in the dynamical systems community), the proof of the genericity of steepness has remained, up to now, largely unstudied and poorly understood. This is certainly due to the fact that such a demonstration does not involve any arguments of dynamical systems, but combines quantitative reasonings of real-algebraic geometry and complex analysis. It is precisely in those reasonings that the Bernstein-Remez inequality plays a major rôle.

11.2.1 The rôle of Bernstein-Remez inequality

A crucial step in Nekhoroshev's proof of the genericity of steepness consists in considering, for any fixed polynomial $P \in \mathbb{R}[X_1,\dots,X_m]$, the semi-algebraic set - called thalweg nowadays (see [28]) - defined by

$$\mathcal{T}_P \subset \mathbb{R}^m := \{u \in \mathbb{R}^m |\ ||\nabla P(u)|| \le ||\nabla P(v)||\ \forall v \in \mathbb{R}^m \text{ s.t. } ||u|| = ||v||\}\ . \quad (11.2.2)$$

Remark 11.2.3. In order to grasp why this kind of set is interesting in the study of the genericity of steepness, it is worth comparing (11.2.2) with (11.2.1) from a heuristic point of view. Infact, in Definition 11.2.1, one is interested in controlling quantitatively the projection of the gradient of the function h on any affine subspace Γ^m which is orthogonal to $\nabla h(I)$. Fixing Γ^m and taking Remark 11.2.1 into account, if one approximates the restriction $h|_{I+\Gamma^m}$ by its Taylor polynomial $P_{h,I+\Gamma^m}$ at a suitable order, then studying the locus

$$\left\{ I+u \in I+\Gamma^m \text{ s.t. } ||\nabla P_{h,I+\Gamma^m}(I+u)|| = \min_{w\in\Gamma^m,\, ||w||=\eta} ||\nabla P_{h,I+\Gamma^m}(I+w)|| \right\}$$

amounts to studying the set $\mathcal{T}_{P_{h,I+\Gamma^m}}$ in (11.2.2), where we have identified $P \equiv P_{h,I+\Gamma^m}$.

[1] A subset of $\mathbb{R}^n$ is said to be semi-algebraic if it can be determined by a finite number of polynomial equalities and inequalities.

Nekhoroshev shows that, for any open ball $\mathcal{B} \subset \mathbb{R}^m$ and for any given polynomial P, the intersection $\mathcal{T}_P \cap \mathcal{B}$ contains a real analytic curve C such that both the distance between the extremities of C and the complex analyticity width of its parametrization admit a lower bound that depends only on m and on the degree of the polynomial P. More specifically, C can be parametrized by algebraic functions. The existence of a uniform Bernstein-Remez inequality (also proved in [94] in a less general context than the one we consider in the following paragraphs) ensures uniform upper bounds on the derivatives of these charts.

The uniform control on the parametrization of the curve C is unavoidable in [94], since it ensures that - for a smooth function - steepness is an open property which can be determined by the Taylor expansion at a certain order (we have a "finite-jet" determinacy of steepness). Namely, with the setting of Definition 11.2.1, if for any m-dimensional subspace Γ^m orthogonal to $\nabla h(I)$ the Taylor polynomial $P_{h,I+\Gamma^m}$ verifies condition (11.2.1), then the uniform control on the derivatives of the curve C contained in the thalweg $\mathcal{T}_{P_{h,I+\Gamma^m}}$ ensures that estimate (11.2.1) is verified uniformly also by polynomials belonging to a neighborhood of $P_{h,I+\Gamma^m}$.

In this way, the study of the genericity of steepness is reduced to the study of uniform lower estimates of the kind (11.2.1) in a finite-dimensional setting which involves polynomials of bounded order. This aspect, together with additional technicalities which will not be discussed here, is crucial in order to prove that the Taylor polynomial of suitably high order of non-steep functions are contained in a semi-algebraic set having positive codimension in the space of polynomials of bounded order. This aspect will be investigated and specified in a forthcoming paper of the first author.

11.3 Rôle in semi-algebraic geometry

Actually, the result about the thalweg described above is a particular case of a general theorem about analytic reparametrizations of semi-algebraic sets. Namely, in refs. [116] and [118], Yomdin has shown that - with the exception of a small part - any two-dimensional semi-algebraic set can be covered by the images of a finite number of real-analytic, algebraic charts of the interval $[-1, 1]$. Moreover, thanks to the existence of a Bernstein-Remez inequality for algebraic functions, one has a bound over the size of all the derivatives of these charts that depends only on the order of the derivation and on the degrees of the polynomials involved in the definition of the considered semi-algebraic set. This is a partial extension of the theorem (called Algebraic Lemma) about the C^k-reparametrization of semi-algebraic sets proved independently by Yomdin and Gromov (see [115], [67], [44]). The analytic reparametrization in [116] result has recently been generalized (see [26] and [50]) to higher dimensional sets with more general structures than semi-algebraic, which allows for important applications in arithmetics.

From a more general point of view, the steepness condition is introduced to prevent the abundance of rational vectors on certain sets. In particular, deep applications of the controlled analytic parametrizations of semi-algebraic sets - yielding bounds on the number of integer points in semi-algebraic sets - are given in [26] and [50]. Along these lines of ideas, the Yomdin-Gromov algebraic lemma with tame parametrizations of semi-algebraic sets (see [115], [67]) was used by Bourgain, Goldstein, and Schlag [37] to bound the number of integer points in a two-dimensional semi-algebraic set.

11.4 Different strategies of proof

In ref. [94], Nekhoroshev proves the existence of a Bernstein-Remez inequality for algebraic functions in his specific problem, by exploiting the properties of complex algebraic curves and by making an intensive use of complex analysis (especially, of compactness arguments exploiting Montel's Theorem). The original statements are difficult to disentangle from the context of the genericity of steepness and their proofs are very sketchy. The existence of Bernstein-Remez inequalities in more general situations has been proved in relatively more recent times by Roytwarf-Yomdin [107], Briskin-Yomdin [38], and Yomdin [117], by combining the controlled growth of the Taylor coefficients of p-valent functions[2] together with arguments of analytic geometry. Moreover, in a closely related problem, Brudnyi has proved in [43] the existence of Bernstein-Remez inequalities for polynomials restricted to graphs of multivariate holomorphic functions.

Nekhoroshev's different strategy of proof is briefly mentioned in [107] (p. 848), without quoting [94]. The strategy of Brudnyi's work [43] relies mainly on potential theory. In particular, Lemma 2.1 in [43] contains a reasoning similar to a minor part of Nekhoroshev's reasonings in combination with a result by Sadullaev (see [109]). However, the overall framework of [43] is very different from Nekhoroshev's one, and the core of Nekhoroshev's arguments does not appear (in particular, Lemma 13.0.2 below). In conclusion, so far we have not been able to find any reference that shows Nekhoroshev's proof in detail except for the original paper (see [94], Lemma 5.1, p.446).

This is our motivation for a short, self-contained exposition of Nekhoroshev's proof relying on arguments complex analysis. Actually, Nekhoroshev [94] shows the existence of a Bernstein-Remez inequality only in the case in which the compact set $\mathcal{K}$ is a real segment and the considered algebraic functions have a particular form, since this is sufficient for his purposes. Here, we extend this strategy by considering any compact set $\mathcal{K}$ of high enough cardinality and we get rid of the additional conditions on the form of the algebraic functions.

Nekhoroshev's approach presents two drawbacks. It does not allow for quantitative estimates for the Bernstein constants as in [107] and [117]. Moreover, we were not able to prove a Bernstein-Remez inequality for an algebraic function on its maximal

[2] An analytic function over a disc is said to be p-valent if either it is constant or each element of $\mathrm{Im}(f)$ is the image of at most k points. Any algebraic function f satisfying $S(z, f(z)) = 0$ for some non-zero polynomial $S \in \mathbb{C}[X, Y]$ of degree k is k-valent (Lemma 12.3.1).

disk of regularity, what is obtained in [107] and is called structural inequality, but only for the maximal disk of regularity of all the algebraic functions associated to the considered polynomial. However, these two points are not mandatory for applications of the Bernstein-Remez inequality to Nekhoroshev's arguments on the thalweg and, more generally, to describe the overall structure of semi-algeraic sets (see [116]).

Finally, as it was already known in [94] and is central in [107], the existence of uniform Bernstein's constants implies uniform bounds on the Taylor coefficients of algebraic functions. In this spirit, we shall also state a result of this kind in Corollary 12.2.2.

11.5 Main result

By the discussion above, it is of crucial importance to find classes of functions admitting a uniform bound on their Bernstein's constants, and thus satisfying a uniform Bernstein-Remez inequality. In this part we will establish the existence of a uniform Bernstein-Remez inequality for the following class of analytic-algebraic functions:

Definition 11.5.1. Consider $k \in \mathbb{N}$, $\rho > 0$ and denote by $\mathcal{D}_\rho(0)$ the open complex disk of radius ρ centered at the origin.

We indicate by $\mathcal{V}(k, \rho)$ the set of functions f that satisfy:

1. f is holomorphic over $\mathcal{D}_\rho(0)$;
2. The graph of f is included in an algebraic curve
$$\mathrm{R}_S := \{(z, w) \in \mathbb{C}^2 : S(z, w) = 0\}$$
associated to a non-zero polynomial $S \in \mathbb{C}[z, w]$ of degree at most k, hence
$$S(z, f(z)) = 0 \qquad \text{for } z \in \mathcal{D}_\rho(0) ;$$
3. The algebraic curve R_S is such that $\mathrm{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$ is the union of at most k elements that can be either vertical lines of the form $\{(z, w) \in \mathbb{C}^2 \mid z = z_*\}$ or disjoint graphs of holomorphic functions over $\mathcal{D}_\rho(0)$.

The functions in the class $\mathcal{V}(k, \rho)$ verify the following

Theorem 11.5.1 (Main result)**.** *With the notations of Definition 11.5.1, consider a compact set $\mathcal{K} \subset \mathcal{D}_\rho(0)$ satisfying:*

$$0 \in \mathcal{K} \textit{ and } \textit{card}\,(\mathcal{K}) > k. \tag{11.5.1}$$

Then, the functions of the family $\mathcal{V}(k, \rho)$ verify a uniform Bernstein-Remez inequality with respect to $\mathcal{K}$ and to any open set Ω such that $\mathcal{K} \subset \Omega$ and $\overline{\Omega} \subset \mathcal{D}_\rho(0)$.

Consequently, there exists a number $\mathtt{C} = \mathtt{C}(k, \rho, \mathcal{K}, \Omega) > 0$ *such that, for any* $f \in \mathcal{V}(k, \rho)$*, one has:*

$$\max_{z \in \overline{\Omega}} |f(z)| \leq \mathtt{C} \max_{z \in \mathcal{K}} |f(z)| \,.$$

This theorem has been demonstrated by Briskin-Yomdin and Roytwarf-Yomdin in refs. [38]- [107] in the cases where $\mathcal{K} = [-\rho', \rho'] \subset \mathbb{R}$ or $\mathcal{K} = \overline{\mathcal{D}}_{\rho'}(0) \subset \mathbb{C}$, and $\Omega = \mathcal{D}_{\rho''}(0) \subset \mathbb{C}$, with $0 < \rho' < \rho'' < \rho$. Moreover, the authors obtain quantitative estimates on the upper bound $\mathtt{C}(k, \rho', \rho'', \mathcal{K})$ for the Bernstein's constant and they generalize these results to relevant cases of algebraic families of holomorphic functions. More recently, these estimates have been extended by Yomdin and Friedland to the case of a discrete compact $\mathcal{K}$ of sufficiently high cardinality in refs. [117] and [63], thanks to the introduction of a geometric invariant related to entropy.

This part is organized as follows: chapter 12 contains the mathematical setting, together with the proof of the main result and of other strictly related statements. Chapter 13 is devoted to the proof of some technical lemmas that are used in chapter 12 and is the "core" of Nekhoroshev's strategy (especially Lemma 13.0.2). Finally, we have relegated to the appendices the statements of some auxiliary results that are used throughout this part.

Chapter 12

Setting, main proof, and auxiliary statements

12.1 Setting

For any $r > 0$ and any $z_0 \in \mathbb{C}$, we denote by $\mathcal{D}_r(z_0)$ the open complex disk centered at z_0 and by $\overline{\mathcal{D}}_r(z_0)$ its closure.

$\mathbb{C}[z, w]$ indicates the ring of polynomials of two variables over the complex field. Throughout this part, we will often identify $\mathbb{C}[z, w]$ with $\mathbb{C}[z][w]$, the ring of complex polynomials in w over the ring of polynomials of the complex variable z.

For $k \in \mathbb{N}$, we indicate by $\mathcal{Q}(k) \subset \mathbb{C}[w]$ and $\mathcal{P}(r, n) \subset \mathbb{C}[z, w]$ respectively the subspaces of complex polynomials in one and two variables having degree inferior or equal to k. Since $\mathcal{Q}(k), \mathcal{P}(r, n)$ are finite-dimensional, they can be equipped with an arbitrary norm.

12.2 Main proof and auxiliary statements

With the notations of Theorem 11.5.1, we consider the following class of functions:

Definition 12.2.1. For $k \in \mathbb{N}$ and $\rho > 0$, we denote by $\mathcal{V}_0(k, \rho) \subset \mathcal{V}(k, \rho)$ the subset of those functions $g \in \mathcal{V}(k, \rho)$ that satisfy $g(0) = 0$.

The functions of the family $\mathcal{V}_0(k, \rho)$ belong to the same Bernstein's class w.r.t. the sets Ω and $\mathcal{K}$ of Theorem 11.5.1. Namely, one has:

Theorem 12.2.1. *Consider an open set Ω satisfying $\overline{\Omega} \subset \mathcal{D}_\rho(0)$ and $\mathcal{K} \subset \Omega$ a compact set satisfying card $\mathcal{K} > k$. There exists a number $\mathtt{C}_0 = \mathtt{C}(k, \rho, \mathcal{K}, \Omega) > 0$ that bounds uniformly the Bernstein's constants of the elements of $\mathcal{V}_0(k, \rho)$, i.e.:*

$$\text{for any } g \in \mathcal{V}_0(k, \rho), \text{ one has } \max_{z \in \overline{\Omega}} |g(z)| \leq \mathtt{C}_0 \max_{z \in \mathcal{K}} |g(z)|.$$

Remark 12.2.1. The hypothesis $0 \in \mathcal{K}$ of Theorem 11.5.1 is unnecessary in Theorem 12.2.1.

Theorem 11.5.1 is a consequence of Theorem 12.2.1 since, for any $f \in \mathcal{V}(k,\rho)$, the function $g(z) := f(z) - f(0)$ belongs to the class $\mathcal{V}_0(k,\rho)$ and Theorem 12.2.1 ensures:

$$\begin{aligned}\max_{\overline{\Omega}} |f| \leq & |f(0)| + \max_{\overline{\Omega}} |g| \leq |f(0)| + \mathtt{C}_0 \max_{\mathcal{K}} |g| \\ & \leq |f(0)| + \mathtt{C}_0 |f(0)| + \mathtt{C}_0 \max_{\mathcal{K}} |f| = (1 + 2\mathtt{C}_0) \max_{\mathcal{K}} |f| := \mathtt{C} \max_{\mathcal{K}} |f|\end{aligned}$$

where the last estimate comes from the hypothesis $0 \in \mathcal{K}$.

This concludes the proof of Theorem 11.5.1.

Theorem 12.2.1 is also the cornerstone which allows one to prove a uniform upper bound on the Taylor coefficients of functions in $\mathcal{V}_0(k,\rho)$. More specifically, we introduce the following class of bounded algebraic functions:

Definition 12.2.2. With the previous notations, for any $M \geq 0$ and any compact $\mathcal{K} \subset \mathcal{D}_\rho(0)$, we denote by $\mathcal{U}(k,\rho,\mathcal{K},M)$ the subset of those functions $g \in \mathcal{V}_0(k,\rho)$ that satisfy $\max_{\mathcal{K}} |g| = M$.

Hence, we have $\mathcal{V}_0(k,\rho) = \cup_{M \geq 0} \mathcal{U}(k,\rho,\mathcal{K},M)$.

The functions in $\mathcal{U}(k,\rho,\overline{\Lambda},L_\Lambda)$ satisfy a generalized uniform Cauchy inequality, namely

Theorem 12.2.2. *Under the additional assumption card $\mathcal{K} > k$, there exists a constant $K = K(k,\rho,\mathcal{K})$ such that, for any function $g \in \mathcal{U}(k,\rho,\overline{\Lambda},L_\Lambda)$, the coefficients of the Taylor series*

$$g(z) = \sum_{j=1}^{+\infty} a_j z^j \quad (with\ g(0) = 0) \tag{12.2.1}$$

satisfy the uniform inequality

1.
$$|a_j| \leq K(k,\rho,\mathcal{K})M \qquad if\ \rho > 1\ ;$$

2. *for any number $m > 1$*
$$|a_j| \leq K(k,m,\mathcal{K})M \left(\frac{m}{\rho}\right)^j \qquad if\ \rho \leq 1\ .$$

Remark 12.2.2. This result is stated and used in [94] in the particular case where $\rho > 1$, $\mathcal{K} = [0,\lambda] \subset \mathbb{R}$, $M(\lambda) = \lambda$ and $\lambda > 0$. The equivalence between a uniform bound on the growth of the Taylor coefficients and the Bernstein-Remez inequality is central in [107].

Theorems 12.2.1 and 12.2.2 will be proved in the next section.

12.3 Proof of the auxiliary statements

We first need the following standard lemma:

Lemma 12.3.1. *With the notations of the previous section, an analytic-algebraic function f, associated to a polynomial $S \in \mathbb{C}[z,w]$ of degree $k \in \mathbb{N}$, is k-valent: that is, if f is not constant then each element of* $\mathrm{Im}(f)$ *is the image of at most k points. Consequently, if f is not identically zero, then f cannot be identically zero over any set $\mathcal{K}$ included in the domain of definition of f such that* $\mathrm{Card}(\mathcal{K}) > k$.

Proof. Assume, by contradiction, that f is non-constant and that there exists $w_0 \in \mathrm{Im}(f)$ which is the image of at least $p > k$ points. The polynomial $S^{w_0}(z) := S(z, w_0)$ would admit $p > k$ roots while $\deg(S^{w_0}) \leq k$ by hypothesis. The Fundamental Theorem of Algebra ensures that S^{w_0} must be identically zero and one has the factorization $S(z,w) = (w - w_0)^\alpha \hat{S}(z,w)$, where $\alpha \in \{1, ..., k\}$, while $\hat{S}$ cannot be divided by $(w - w_0)$ in $\mathbb{C}[z,w]$. Since f is analytic and not constant, the preimage $f^{-1}(\{w_0\})$ is a discrete set and the graph of f must satisfy $\hat{S}(z, f(z)) = 0$ outside of $f^{-1}(\{w_0\})$. By continuity, one has $\hat{S}(z, f(z)) = 0$ on the whole domain of definition of f since $f^{-1}(\{w_0\})$ is discrete. But $\deg \hat{S}^{w_0} \leq k$, with $\hat{S}^{w_0}(z) := \hat{S}(z, w_0)$, and $\hat{S}^{w_0}$ admits more than k roots, hence the previous argument ensures that $\hat{S}$ can be divided by $(w - w_0)$, in contradiction to construction.

Moreover, if $f \not\equiv 0$, then 0 admits at most k inverse images by f, and f cannot be identically null over any set $\mathcal{K}$ included in the domain of definition of f satisfying $\mathrm{card}\, \mathcal{K} > k$. □

Consequently, without any loss of generality, in Theorem 12.2.1 we can assume $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ according to Definition 12.2.2 (hence $g \in \mathcal{V}_0(k, \rho)$ and $\max_{\mathcal{K}} |g| = 1$) since, if this is not the case, it suffices to consider $g / \max_{\mathcal{K}} |g|$.

Then, we define the following set:

Definition 12.3.1. $\mathcal{A} := \mathcal{A}(\mathcal{K}, k, \rho)$ denotes the set of those polynomials $S \in \mathcal{P}(r, n) \backslash \{0\}$ whose algebraic curve $\mathrm{R}_S := \{(z,w) \in \mathbb{C}^2 : S(z,w) = 0\}$ satisfies

1. $\mathrm{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$ is the union of at most k elements that can be either vertical lines of the form $\{(z,w) \in \mathbb{C}^2 \mid z = z_*\}$ or disjoint graphs of holomorphic functions over $\mathcal{D}_\rho(0)$;

2. there exists $g_S \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ whose graph is contained in $\mathrm{R}_S \cap \{\mathcal{D}_\rho(0) \times \mathbb{C}\}$.

Remark 12.3.1. For any $S \in \mathcal{A}$, the function g_S is unique since the graphs contained in the algebraic curve of S are disjoint over $\mathcal{D}_\rho(0)$ and the value $g_S(0) = 0$ is fixed.

The central property in the proof of Theorem 12.2.1 is the following

Lemma 12.3.2. *$\mathcal{A} \cup \{0\}$ is closed in $\mathcal{P}(r, n)$ and, for any open set Ω satisfying $\mathcal{K} \subset \Omega$, $\overline{\Omega} \subset \mathcal{D}_\rho(0)$, the function*

$$\mu_\Omega : \mathcal{A} \longrightarrow \mathbb{R} \qquad S \longmapsto \max_{\overline{\Omega}} |g_S|$$

is continuous.

We shall relegate the proof of Lemma 12.3.2 to the next section and we shall exploit its statement here to prove Theorems 12.2.1 and 12.2.2.

Proof. (Theorem 12.2.1)
By Definitions 12.2.1, 12.2.2 and 12.3.1, we can associate to any $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ a polynomial $S \in \mathcal{A}$ such that $g = g_S$. A standard combinatorial computation yields that $\mathcal{P}(r, n)$ is isomorphic to $\mathbb{C}^m$, with $m = (k + 1)(k + 2)/2$. It is also easy to see that for any polynomial $S \in \mathcal{A}$ and for any $c \in \mathbb{C}\backslash\{0\}$ the polynomial $S' = cS$ belongs to $\mathcal{A}$ and $g_{S'} \equiv g_S$, so that it makes sense to pass to the projective space

$$\mathbb{CP}^{m-1} := \{\mathbb{C}^m\backslash\{0\}\}/\{\mathbb{C}\backslash\{0\}\} \quad , \qquad \pi : \mathbb{C}^m\backslash\{0\} \longrightarrow \mathbb{CP}^{m-1} \ ,$$

where π denotes the standard canonical projection inducing the quotient topology in $\mathbb{CP}^{m-1}$. Moreover, for any open set Ω satisfying $\mathcal{K} \subset \Omega$, $\overline{\Omega} \subset \mathcal{D}_\rho(0)$, the function

$$\hat{\mu}_\Omega : \pi(\mathcal{A}) \longrightarrow \mathbb{R} \quad , \qquad \pi(S) \longmapsto \max_{\overline{\Omega}} |g_S|$$

is well defined and continuous by Lemma 12.3.2. To prove the latter claim, take a closed set $\mathcal{E} \subset \mathbb{R}$ and consider its inverse image $\hat{\mu}_\Omega^{-1}(\mathcal{E}) = \pi(\mu_\Omega^{-1}(\mathcal{E}))$. Since μ_Ω is continuous, $\mu_\Omega^{-1}(\mathcal{E})$ is closed in $\mathcal{A}$ for the induced topology. By Lemma 12.3.2, $\mathcal{A} \cup \{0\}$ is closed in $\mathbb{C}^m$, so that $\mathcal{A}$ is closed in $\mathbb{C}^m\backslash\{0\}$. Hence, $\mu_\Omega^{-1}(\mathcal{E})$ is closed in $\mathbb{C}^m\backslash\{0\}$. Since μ_Ω is invariant if its argument is multiplied by a complex non-zero constant, $\mu_\Omega^{-1}(\mathcal{E})$ is saturated and one has $\pi^{-1}(\pi(\mu_\Omega^{-1}(\mathcal{E}))) = \mu_\Omega^{-1}(\mathcal{E})$. Consequently, the set $\pi(\mu_\Omega^{-1}(\mathcal{E})) = \hat{\mu}_\Omega^{-1}(\mathcal{E})$ is closed for the quotient topology because its inverse image w.r.t. π is closed. This proves the continuity of $\hat{\mu}_\Omega$.

Moreover, since $\mathcal{A}$ is closed and saturated in $\mathbb{C}^m\backslash\{0\}$, $\pi(\mathcal{A})$ is closed in $\mathbb{CP}^{m-1}$ and the compactness of $\mathbb{CP}^{m-1}$ ensures that $\pi(\mathcal{A})$ is compact. By continuity of $\hat{\mu}_\Omega$, the image $\hat{\mu}_\Omega(\pi(\mathcal{A}))$ is a compact subset of $\mathbb{R}$, hence bounded. Therefore, there exists a constant $\mathtt{C}(k, \rho, \mathcal{K}, \Omega)$ such that for any $g \in \mathcal{U}(k, \rho, \mathcal{K}, 1)$ one has

$$\max_{\overline{\Omega}} |g| = \frac{\max_{\overline{\Omega}} |g|}{\max_{\mathcal{K}} |g|} \leq \mathtt{C}(k, \rho, \mathcal{K}, \Omega)$$

and this concludes the proof. □

Proof. (Theorem 12.2.2)
Since g is non identically zero over $\mathcal{K}$ (see Lemma 12.3.1), we can consider the function g/M and we are reduced to the case $M = 1$.

For $\rho > 1$, the statement is a consequence of the Cauchy's estimate and of Theorem 12.2.1 applied to $\Omega = \mathcal{D}_1(0)$ and $\mathcal{K}$.

In case $\rho \leq 1$, for any fixed $m > 1$ one considers the function

$$g_m(z) := g\left(\frac{\rho}{m} z\right) := \sum_{j=1}^{+\infty} c_j z^j = \sum_{j=1}^{+\infty} a_j \left(\frac{\rho}{m} z\right)^j$$

analytic in $\mathcal{D}_m(0)$ and belonging to $\mathcal{U}(k, m, \mathcal{K}_m, 1)$, where

$$\mathcal{K}_m := \{z \in \mathcal{D}_m(0) : \frac{\rho}{m} z \in \mathcal{K}\}$$

satisfies card $\mathcal{K}_m > k$ since $\mathcal{K}$ does.

Since the convergence radius of g_m is $m > 1$, the statement holds for this function and there exists a constant $K(k, m, \mathcal{K})$ such that

$$|c_j| \leq K(k, m, \mathcal{K}) \qquad \forall\, j \in \mathbb{N},$$

which implies

$$|a_j| \leq K(k, m, \mathcal{K}) \left(\frac{m}{\rho}\right)^j .$$

This concludes the proof. □

Chapter 13

Technical lemmas

The aim of this section is to prove Lemma 12.3.2. We first recall a few classical points.

The algebraic curve of a polynomial $S \in \mathbb{C}[z, w]$ is the zero-set

$$\mathtt{R}_S := \{(z, w) \in \mathbb{C}^2 \,:\, S(z, w) = 0\} \,.$$

and one has the following standard result

Lemma 13.0.1. *For any integer $k \geq 1$ and for any polynomial $S \in \mathcal{P}(r, n)$, there exists a set $\mathcal{N}_S \subset \mathbb{C}$ (defined explicitly in Appendix A, see D.0.1) satisfying card $\mathcal{N}_S \leq \mathtt{N}_k$ - where $\mathtt{N}_k \in \mathbb{N}$ is an upper bound depending only of k - and such that over any simply connected domain $\mathtt{D} \subset \mathbb{C}$ the intersection of the algebraic curve $\mathtt{R}_S$ with $\mathtt{D} \times \mathbb{C}$ is the union of at most k disjoint graphs of holomorphic functions defined over $\mathtt{D}$ if and only if $\mathtt{D} \cap \mathcal{N}_S = \emptyset$.*

The proof of this result can be found by putting together known results on algebraic curves (see e.g. [92]). For the sake of clarity, it is given in appendix D.

Remark 13.0.1. Following ref. [94], the elements of $\mathcal{N}_S$ are called *excluded points*.

Remark 13.0.2. The number of graphs in Lemma 13.0.1 may be equal to zero, for example if $S(z, w) = z$, we have $\mathtt{R}_S = \{(z, w) \in \mathbb{C}^2 \,:\, z = 0\}$ and the point $z = 0$ is excluded by construction (see Appendix D).

Definition 13.0.1 (Riemann branches and leaves)**.** In the setting of Lemma 13.0.1, if $\mathtt{R}_S$ is non-empty over $\mathtt{D}$, the holomorphic functions whose graphs cover $\mathtt{D}$ are algebraic since their graphs solve the equation $S(z, w) = 0$ for all $z \in \mathtt{D}$. These functions will henceforth be called the *Riemann branches* of S over $\mathtt{D}$, whereas their graphs will be referred to as the *Riemann leaves* of S over $\mathtt{D}$.

It is a standard fact that, up to constant multiplicative factors, any polynomial $S \in \mathcal{P}(r, n)$ can be uniquely factorized as

$$S(z, w) = q(z)\, \Pi_{i=1}^{m} (S_i(z, w))^{j_i} \tag{13.0.1}$$

for some $1 \le j_i \le k$, $1 \le m \le k$, where the $\mathcal{S}_i$'s are non-constant, irreducible, mutually non-proportional polynomials. Hence, without any loss of generality, we can pass to the unit sphere in $\mathcal{Q}(k)$ and assume $||q|| = 1$ for an arbitrary norm $|| \cdot ||$.

We denote

$$\overline{\mathcal{S}}(z, w) := \Pi_{i=1}^{m} (\mathcal{S}_i(z, w))^{j_i} \tag{13.0.2}$$

and we have the polynomial product:

$$\mathcal{S}(z, w) = q(z)\overline{\mathcal{S}}(z, w). \tag{13.0.3}$$

We start by giving the following

Definition 13.0.2. We indicate by $\mathcal{B} = \mathcal{B}(k, \rho) \subset \mathcal{P}(r, n)$ the set of polynomials $\mathcal{S} \in \mathcal{P}(r, n) \backslash \{0\}$ such that the polynomial $\overline{\mathcal{S}}$ in decomposition (13.0.3) has no excluded points (Definition D.0.1) in $\mathcal{D}_\rho(0)$.

Remark 13.0.3. Given $\mathcal{S} \in \mathcal{B}$, by decomposition (13.0.3) and Definition D.0.1, the only possible excluded points for $\mathcal{S}$ in $\mathcal{D}_\rho(0)$ are those at which $q(z) = 0$. Inside the disk $\mathcal{D}_\rho(0)$, the algebraic curve $\mathtt{R}_\mathcal{S}$ is therefore the union of at most k elements that can be either disjoint holomorphic Riemann leaves of $\overline{\mathcal{S}}$ or vertical lines in $\mathbb{C}^2$ of the kind $z = z_0$, with $q(z_0) = 0$. In particular, all the Riemann branches of $\mathcal{S} \in \mathcal{B}$ are holomorphic over $\mathcal{D}_\rho(0)$.

Remark 13.0.4. The set $\mathcal{A}$ of Definition 12.3.1 is contained in $\mathcal{B}$ and, with the notations of Theorem 11.5.1, the functions in $\mathcal{V}(k, \rho)$ are precisely those associated to the polynomials in $\mathcal{B}$.

In order to prove Lemma 12.3.2, we need the following

Lemma 13.0.2. *$\mathcal{B} \cup \{0\}$ is closed in $\mathcal{P}(r, n)$.*

The proof of Lemma 13.0.2 is quite technical and requires some intermediate results, which are stated in the sequel.

We start by considering a sequence $\{\mathcal{S}_n(z, w)\}_{n\in\mathbb{N}}$ of polynomials in $\mathcal{B} \cup \{0\}$, converging to a polynomial $\mathcal{S} \in \mathcal{P}(r, n)$. We can assume that $\mathcal{S} \not\equiv 0$ otherwise there is nothing to prove; hence we have $\mathcal{S}_n \not\equiv 0$ for n large enough.

Following decomposition (13.0.3), we write $\mathcal{S}_n(z, w) := q_n(z)\overline{\mathcal{S}}_n(z, w)$ and, by construction, the sequence of polynomials $\{q_n\}_{n\in\mathbb{N}}$ is in the compact unit sphere and admits a convergent subsequence. With slight abuse of notation, in the sequel we shall indicate this subsequence with the same symbol $\{q_n\}_{n\in\mathbb{N}}$ and we shall denote by $\widehat{q}$ its limit, which is not identically null by construction.

We recall that $\mathcal{N}_\mathcal{S}$ and $\mathcal{N}_{\mathcal{S}_n}$ (for $n \in \mathbb{N}$) denote the sets of excluded points of $\mathcal{S}$ and $\mathcal{S}_n$, respectively. For $r > 0$ small enough, we remove from $\mathcal{D}_\rho(0)$ all open neighborhoods of radius r around the excluded points of $\mathcal{S}$ and consider the following compact set:

$$\mathtt{E}_r := \{z \in \overline{\mathcal{D}}_{\rho-r}(0) \;/\; |z - z_0| \ge r \text{ for } z_0 \in \mathcal{N}_\mathcal{S}\} \subset \mathcal{D}_\rho(0). \tag{13.0.4}$$

Lemma 13.0.3. *There exists $r_0 = r_0(\rho, k)$ such that, for any $0 < r \le r_0$, one has $\mathtt{E}_r \neq \varnothing$ and there exists an integer $n_0 = n_0(r)$ such that:*

$$\mathtt{E}_r \cap \mathcal{N}_{S_n} = \varnothing \text{ for all } n \ge n_0 \,. \tag{13.0.5}$$

Proof. The fact that $\mathtt{E}_r \neq \varnothing$ for r sufficiently small is an immediate consequence of Definition 13.0.4 and of the fact that card $\mathcal{N}_S$ is bounded by a number depending only on k (see Lemma 13.0.1).

As for the second part of the statement, since $S_n \longrightarrow S \in \mathcal{P}(r,n)$, and $q_n \to \widehat{q} \not\equiv 0$, there exists a polynomial $\widehat{S} \in \mathcal{P}(r,n)$ such that

$$\lim_{n\longrightarrow+\infty} S_n(z,w) = \lim_{n\longrightarrow+\infty} \overline{S}_n(z,w) \times \lim_{n\longrightarrow+\infty} q_n(z) = \widehat{S}(z,w) \times \widehat{q}(z). \tag{13.0.6}$$

By applying again decomposition (13.0.3) to $\widehat{S}$ we obtain $\widehat{S}(z,w) = \widetilde{q}(z)\overline{S}(z,w)$, so that we can write $S(z,w) = q(z)\overline{S}(z,w)$ by setting

$$q(z) := \widehat{q}(z) \times \widetilde{q}(z) \,. \tag{13.0.7}$$

Therefore, all the roots of $\widehat{q}$ are also roots of q and belong to $\mathcal{N}_S$. By construction (see also remark 13.0.3), all points in $\mathcal{N}_{S_n}$ are roots of $q_n(z) = 0$. Since $q_n \longrightarrow \widehat{q}$, taking into account the continuous dependence of the roots of a polynomial on its coefficients expressed in Theorem F.0.1, one has that for sufficiently high n the roots of q_n must be either r-close to the roots of $\widehat{q}$, and hence to some point of $\mathcal{N}_S$, or outside of the disc of radius $\mathcal{D}_{1/r}(0)$. Taking $r_0 < 1/\rho$, one has $\mathcal{D}_{1/r}(0) \supset \mathcal{D}_\rho(0)$, whence the conclusion. □

We fix $0 < r \le r_0$, with r_0 the bound in Lemma 13.0.3, and we consider a point $z^\star \in \mathtt{E}_r$, hence $z^\star$ is not an excluded point of S and any solution of $S^{z^\star}(w) := S(z^\star, w) = 0$ must belong to the image of a Riemann branch of S holomorphic in a neighbourhood of $z^\star$. We fix one of these branches and denote it with h. The continuous dependence of the zeros of a polynomial on its coefficients (Theorem F.0.1) ensures the existence of a sequence $\{w_n^\star\}_{n\in\mathbb{N}}$ of roots of $S_n^{z^\star}(w) := S_n(z^\star, w)$ such that

$$w_n^\star \longrightarrow h(z^\star) \,.$$

Lemma 13.0.3 and Remark 13.0.3 together with the fact that $S_n \in \mathcal{B}$ for all $n \in \mathbb{N}$ ensure that, for any fixed $n > n_0(r)$, the point $(z^\star, w_n^\star)$ must belong to the Riemann leaf of one of the branches of $\overline{S}_n$, denoted h_n, which is holomorphic over $\mathcal{D}_\rho(0)$. Hence we have the pointwise convergence

$$h_n(z^\star) \longrightarrow h(z^\star) \,. \tag{13.0.8}$$

We show in the sequel that the sequence $\{h_n\}_{n\in\mathbb{N}}$ admits a subsequence that converges uniformly on any compact subset of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a holomorphic function which extends h over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$. In order to prove this claim, which is fundamental to the proof of Lemma 13.0.2, we need the following results.

Lemma 13.0.4. *The Riemann branches of S are bounded on the compact sets included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. By construction, any point $\hat{z} \in \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ is regular for S, hence there exists an open neighbourhood $V \subset \mathbb{C}$ of $\hat{z}$ such that the algebraic curve $\mathtt{R}_S \cap \{V \times \mathbb{C}\}$ is composed of at most k graphs of holomorphic functions bounded over V. Since any compact set included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ can be covered by a finite number of these neighbourhoods, the claim is proved. □

Lemma 13.0.5. *The sequence $\{h_n\}_{n\in\mathbb{N}}$ is locally bounded (E.0.2) over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. If, by contradiction, there exists a compact $\mathtt{K} \subset \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ such that $\{h_n\}_{n\in\mathbb{N}}$ is unbounded over $\mathtt{K}$, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathtt{K}$ and a strictly increasing function φ over $\mathbb{N}$ such that the subsequence $\{|h_{\varphi(n)}(z_n)|\}_{n\in\mathbb{N}}$ diverges.

By Definition (13.0.4), $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S = \cup_{r>0}\mathtt{E}_r$, so there exists $0 < r \le r_0$ small enough such that $\mathtt{K} \subset \mathtt{E}_r \subset \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$. Moreover, $\mathtt{E}_r$ is a compact, arc-connected set since it is $\overline{\mathcal{D}}_{\rho-r}(0)$ without a finite number of open disks. Then, for any $n \in \mathbb{N}$ it is always possible to construct a continuous arc:

$$\gamma_n : [0,1] \longrightarrow \mathtt{E}_r \text{ with } \gamma_n(0) = z^\star \text{ and } \gamma_n(1) = z_n.$$

We introduce the continuous functions:

$$\psi_n : [0,1] \longrightarrow \mathbb{R} \quad , \qquad \psi_n(t) := |h_{\varphi(n)}(\gamma_n(t))| \; .$$

Since $S_n^z \to S^z$ uniformly for $z \in \overline{\mathcal{D}}_{\rho-r}(0)$, Theorem F.0.1 ensures that, for all $\varepsilon > 0$, there exists $\mathtt{n}(\varepsilon) \in \mathbb{N}$ such that for all $n > \mathtt{n}(\varepsilon)$ and all $z \in \overline{\mathcal{D}}_{\rho-r}(0)$, the roots of S_n^z are either ε-close to the roots of S^z or in the complement of the closed disk $\overline{\mathcal{D}}_{1/\varepsilon}(0)$. Moreover, Lemma 13.0.4 ensures that the roots of S^z are uniformly bounded for all $z \in \mathtt{E}_r$. We indicate by $w_{max}(r)$ the maximal module that the Riemann branches of S can reach on $\mathtt{E}_r$ and we set

$$\varepsilon_0(r) = \frac{1}{w_{max}(r)+1}.$$

In this setting, we can consider a fixed integer $n > \mathtt{n}(\varepsilon_0(r))$ such that:

$$\psi_n(1) > \frac{1}{\varepsilon_0} = w_{max}(r) + 1 \tag{13.0.9}$$

and taking (13.0.8) into account, we also assume that n is high enough to ensure:

$$|\,\psi_n(0) - |h(z^\star)|\,| = |\,|h_{\varphi(n)}(z^\star)| - |h(z^\star)|\,| \le |h_{\varphi(n)}(z^\star) - h(z^\star)| < \varepsilon_0 < 1 \tag{13.0.10}$$

hence $\psi_n(0) < |h(z^\star)| + 1 \le w_{max}(r) + 1$.

With (13.0.9) and (13.0.10), the intermediate value theorem applied to ψ_n implies that there exists $t_\star \in]0,1[$ satisfying

$$\psi_n(t_\star) = w_{max}(r) + 1 \; . \tag{13.0.11}$$

But $n > \mathtt{n}(\varepsilon_0(r))$ and $\gamma_n(t_\star) \in \overline{\mathcal{D}}_{\rho-r}(0)$, hence $h_{\varphi(n)}(\gamma_n(t_\star))$ is either in the complement of the closed disk $\overline{\mathcal{D}}_{1/\varepsilon_0}(0)$ and

$$\psi_n(t_\star) > \frac{1}{\varepsilon_0} = w_{max}(r) + 1 \tag{13.0.12}$$

or ε_0-close to the roots of S^z and

$$\psi_n(t_\star) < w_{max}(r) + \varepsilon_0 < w_{max}(r) + 1. \tag{13.0.13}$$

Conditions (13.0.11), (13.0.12) and (13.0.13) are in contradiction and the statement is proved. □

Lemma 13.0.6. *There exists a subsequence of $\{h_n\}_{n\in\mathbb{N}}$ which converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a Riemann branch of S (still denoted h) extending holomorphically h over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$.*

Proof. With Lemma 13.0.5 and Montel's Theorem E.0.1, it is possible to extract a subsequence - still denoted $\{h_n\}_{n\in\mathbb{N}}$ with slight abuse of notation - that converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a function holomorphic over $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ which is also still denoted h. Finally, thanks to Lemma F.0.1 and to the fact that $S_n \longrightarrow S$, one has $S(z, h(z)) = 0$ for any $z \in \mathcal{D}_\rho(0)$. □

Remark 13.0.5. By the above Lemma, $\mathcal{N}_S$ does not contain any ramification points.

With the help of Lemma 13.0.6, we are now able to prove Lemma 13.0.2.

Proof. (Lemma 13.0.2)

The aim is to prove that the set of excluded points $\mathcal{N}_{\overline{S}}$ for $\overline{S}$ associated to the limit polynomial S is empty, from which the conclusion follows.

Assume that $z_0 \in \mathcal{N}_{\overline{S}}$. Since $\mathcal{N}_{\overline{S}}$ is a finite set, for $t > 0$ small enough the punctured disc $\dot{\mathcal{D}}_t(z_0) := \{z \in \mathcal{D}_\rho(0) : 0 < |z - z_0| < t\}$ is included in $\mathcal{D}_\rho(0)\backslash\mathcal{N}_{\overline{S}}$ and any branch h of the polynomial $\overline{S}$ is holomorphic in $\dot{\mathcal{D}}_t(z_0)$. Then, by Laurent's Theorem E.0.2 and Proposition F.0.1, z_0 is either a removable singularity or a pole. We show that the second possibility does not occur.

If by contradiction z_0 is a pole for h, then $\lim_{z\longrightarrow z_0} h(z)$ is infinite and one can choose the radius t small enough so that $h(z) \neq 0$ for all $z \in \dot{\mathcal{D}}_t(z_0)$. Hence the function $\phi := 1/h$ is analytic on the punctured disc $\dot{\mathcal{D}}_t(z_0)$ and it is also bounded since its limit is zero when z goes to z_0. By Riemann's Theorem (E.0.4) on removable singularities, ϕ admits a holomorphic extension, still denoted ϕ, on the whole disc $\mathcal{D}_t(z_0)$ satisfying $\phi(z_0) = 0$.

Lemma 13.0.6 ensures that there exists a subsequence $\{h_{n_j}\}_{j\in\mathbb{N}}$ of Riemann branches for S_{n_j} (actually, by Remark 13.0.3, the branches h_{n_j} are analytic over $\mathcal{D}_\rho(0)$ since $S_{n_j} \in \mathcal{B}$) which converges uniformly to h on the compact subsets of $\dot{\mathcal{D}}_t(z_0) \subset \mathcal{D}_\rho(0)\backslash\mathcal{N}_S$. Consequently, the functions h_{n_j} do not vanish on any compact subset of the disk $\mathcal{D}_t(z_0)$

for j large enough. This ensures that the functions $\{\phi_{n_j}\}_{j\in\mathbb{N}} := \{1/h_{n_j}\}_{j\in\mathbb{N}}$ are holomorphic on $\mathcal{D}_t(z_0)$. Moreover, the sequence $\{\phi_{n_j}\}_{j\in\mathbb{N}}$ converges locally uniformly to ϕ on $\dot{\mathcal{D}}_t(z_0)$. Since both ϕ_{n_j} and ϕ are holomorphic at z_0, by the Maximum Principle, this convergence is actually locally uniform over the whole disc $\mathcal{D}_t(z_0)$.

On the one hand, we have $\phi(z_0) = 0$ and $\phi(z) \neq 0$ for $z \in \dot{\mathcal{D}}_t(z_0)$, since in this domain $\phi(z) = 1/h(z)$ and h is holomorphic on $\dot{\mathcal{D}}_t(z_0)$.

On the other hand, the terms of the subsequence $\{\phi_{n_j}\}_{j\in\mathbb{N}} := \{1/h_{n_j}\}_{j\in\mathbb{N}}$ are nowhere-vanishing on $\mathcal{D}_t(z_0)$ and ϕ_{n_j} is holomorphic in that domain. Consequently, by Hurwitz's Theorem E.0.5 on sequences of holomorphic functions, ϕ must be either identically zero or nowhere null on $\mathcal{D}_t(z_0)$.

We have obtained a contradiction and therefore $\lim_{z\longrightarrow z_0} h(z)$ is finite. By applying once again Riemann's Theorem on removable singularities, h admits an analytic extension $\widetilde{h}$ to the whole disc $\mathcal{D}_t(z_0)$. Moreover, $\widetilde{h}$ is a Riemann branch of $\overline{S}$ in the whole disc $\mathcal{D}_t(z_0)$, since

$$\overline{S}(z_0, \widetilde{h}(z_0)) = \lim_{z\longrightarrow z_0} \overline{S}(z, h(z)) = 0\ .$$

It remains to rule out the possibility that z_0 is singular because the graphs of two distinct branches h and ℓ of the limit polynomial $\overline{S}$ intersect on it. Assume that $h(z_0) = \ell(z_0)$. By Lemma 13.0.6 and by the previous arguments, there exist two subsequences $\{h_{n_j}\}_{j\in\mathbb{N}}$ and $\{\ell_{n_j}\}_{j\in\mathbb{N}}$ of branches associated to $\{\overline{S}_{n_j}\}_{j\in\mathbb{N}}$ that approach respectively h and ℓ locally uniformly over $\mathcal{D}_t(z_0)$.

We first notice that h_{n_j} is distinct from ℓ_{n_j} for j large enough, otherwise there exists a subsequence of common branches $h_{n_j}{=}\ell_{n_j}$ for $\overline{S}_{n_j}$ up to infinity which converges locally uniformly in $\mathcal{D}_t(z_0)$ respectively to h and ℓ. Consequently $h = \ell$ over $\mathcal{D}_t(z_0)$, which contradicts the assumption that h and ℓ are distinct. Moreover, since $\mathrm{R}_{\overline{S}_{n_j}}$ is composed of distinct regular leaves over $\mathcal{D}_t(z_0)$ for any $j \in \mathbb{N}$, the functions $h_{n_j}{-}\ell_{n_j}$ never vanish over $\mathcal{D}_t(z_0)$.

Consequently, Hurwitz's Theorem E.0.5 ensures that the sequence of holomorphic functions $\{h_{n_j}{-}\ell_{n_j}\}_{j\in\mathbb{N}}$ converges to a limit which either never vanishes or is identically zero over $\mathcal{D}_t(z_0)$. Here $\lim_{j\to+\infty}(h_{n_j}{-}\ell_{n_j})(z_0) = (h - \ell)(z_0) = 0$, so $h = \ell$ everywhere over $\mathcal{D}_t(z_0)$, which is again in contradiction with the assumption that h and ℓ are distinct.

Therefore, we have proved that the algebraic curve of the limit polynomial $\overline{S}$ is composed of disjoint and regular Riemann leaves over a neighborhood of z_0. Since the above arguments hold for any $z_0 \in \mathcal{N}_S$, the algebraic curve $\mathrm{R}_{\overline{S}}$ is composed of distinct leaves over $\mathcal{D}_\rho(0)$ and the branches of $\overline{S}$ are globally holomorphic over $\mathcal{D}_\rho(0)$, consequently $\mathcal{N}_{\overline{S}} = \varnothing$ (see Remark 13.0.3). □

Lemma 13.0.2 is the cornerstone for the proof of Lemma 12.3.2.

Proof. (Lemma 12.3.2)

We start by proving the closure of $\mathcal{A} \cup \{0\}$ in $\mathcal{P}(r,n)$ and consider a sequence $\{S_n\}_{n\in\mathbb{N}}$ in $\mathcal{A} \cup \{0\}$ which converges to a limit $S \in \mathcal{P}(r,n)$. One has $S \in \mathcal{B} \cup \{0\}$, since $\mathcal{A} \cup \{0\} \subset \mathcal{B} \cup \{0\}$ and $\mathcal{B} \cup \{0\}$ is closed by Lemma 13.0.2.

By hypothesis, for any fixed $n \in \mathbb{N}$ there exists a Riemann branch $g_n(z)$ which is analytic on $\mathcal{D}_\rho(0)$ and satisfies

$$S_n(z, g_n(z)) = 0\ ,\quad g_n(0) = 0\ ,\quad \max_{z\in\mathcal{K}} |g_n(z)| = 1\ . \tag{13.0.14}$$

If $S \equiv 0$ there is nothing to prove.

If $S \not\equiv 0$, we claim that $\{g_n\}_{n\in\mathbb{N}}$ has a subsequence that converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)$ to a branch g_S of S having the desired properties.

In fact, since $S \in \mathcal{B} \cup \{0\}$, the elements of the set $\mathcal{N}_S$ are the roots of $q(z) = 0$. Consequently, card $\mathcal{N}_S \leq k$ and card $\mathcal{K} > k$ ensures that there exists $z^\star \in \mathcal{K}\backslash\mathcal{N}_S$ such that $\{g_n(z^\star)\}_{n\in\mathbb{N}}$ is bounded. Up to the extraction of a subsequence, $g_n(z^\star)$ converges to a complex value $w^\star$. Moreover, since $z^\star$ is not an excluded point of $S \in \mathcal{B}$, we can ensure that $(z^\star, w^\star)$ belongs to a Riemann leaf of $\mathtt{R}_{\overline{S}}$ which is associated to a holomorphic Riemann branch over $\mathcal{D}_\rho(0)$, denoted g_S.

By the same arguments used in the proofs of Lemmas 13.0.5 and 13.0.6, the sequence $\{g_n\}_{n\in\mathbb{N}}$ admits a subsequence which converges uniformly on the compact subsets of $\mathcal{D}_\rho(0)\backslash\mathcal{N}_S$ to a Riemann branch f_S associated to S. The holomorphy of f_S over $\mathcal{D}_\rho(0)$ (since $S \in \mathcal{B}$) and the Maximum Principle imply that the convergence is actually locally uniform on the whole set $\mathcal{D}_\rho(0)$. Then, by the uniqueness of the limit, we have $g_S(z^\star) = f_S(z^\star)$, which implies $g_S \equiv f_S$ over $\mathcal{D}_\rho(0)$ because $S \in \mathcal{B}$. This yields $\max_{\mathcal{K}} |g_S| = 1$ and $g_S(0) = 0$, hence g_S meets the requirements of Definition 12.3.1 and $S \in \mathcal{A}$.

Finally, it remains to prove that the function μ_Ω in Lemma 12.3.2 is continuous. Since $\{g_n\}_{n\in\mathbb{N}}$ converges locally uniformly to g_S in $\mathcal{D}_\rho(0)$, we can write

$$\lim_{n\longrightarrow+\infty} \left| \max_{z\in\mathcal{K}'} |g_S(z)| - \max_{z\in\mathcal{K}'} |g_n(z)| \right| \leq \lim_{n\longrightarrow+\infty} \left(\max_{z\in\mathcal{K}'} |g_S(z) - g_n(z)| \right) = 0\ , \tag{13.0.15}$$

for any compact $\mathcal{K}' \subset \mathcal{D}_\rho(0)$. By taking $\mathcal{K}' \equiv \overline{\Omega} \subset \mathcal{D}_\rho(0)$, we have

$$\mu_\Omega(S) := \max_{z\in\overline{\Omega}} |g_S(z)| = \lim_{n\longrightarrow+\infty} \left(\max_{z\in\overline{\Omega}} \left|g_n(z)\right| \right) =: \lim_{n\longrightarrow+\infty} \left(\mu_\Omega(S_n) \right),$$

which implies that μ_Ω is continuous. This concludes the proof of Lemma 12.3.2.

□

Part III

Analytic Smoothing and Nekhoroshev estimates for Hölder steep Hamiltonians

Abstract

In this part we prove the first result of Nekhoroshev stability for steep Hamiltonians in Hölder class. Our new approach combines the classical theory of normal forms in analytic category with an improved smoothing procedure to approximate an Hölder Hamiltonian with an analytic one. It is only for the sake of clarity that we consider the (difficult) case of Hölder perturbations of an analytic integrable Hamiltonian, but our method is flexible enough to work in many other functional classes, including the Gevrey one. The stability exponents can be taken to be $(\ell-1)/(2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-2})+1/2$ for the time of stability and $1/(2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-1})$ for the radius of stability, n being the dimension, $\ell > n+1$ being the regularity and the $\boldsymbol{\alpha}_i$'s being the indices of steepness. Crucial to obtain the exponents above is a new non-standard estimate on the Fourier norm of the smoothed function. As a byproduct we improve the stability exponents in the C^k class, with integer k.

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Chapter 14

Introduction and main results

The main goal of this work is to introduce a unified way for proving "long time stability" of the action variables for perturbations of completely integrable Hamiltonian systems which belong to a large class of function spaces. We will limit ourselves here to Hölder perturbations of analytic systems, but our method is flexible enough to be adapted to many other settings[1].

The effective stability theory for nearly-integrable Hamiltonian systems was initiated by the pioneering work of J.E. Littlewood [82] and reached a first main achievement in the seventies with the work of N.N. Nekhoroshev [95]; it was then developed by many authors. The usual setting is that of Hamiltonian systems of the form

$$H(I, \theta) = h(I) + f(I, \theta), \tag{14.0.1}$$

where $(I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ are the action-angle variables and f is small with respect to h. In Nekhoroshev's work the Hamiltonian H is analytic and h satisfies a steepness condition (see definition 14.3.1 below). The theory has been then developed in various settings: H can be assumed to be Gevrey (which includes the analytic case) or C^k with $k \geq 2$ and integer, while h can be assumed to be convex or quasi-convex (see for example [87] or [30])

The norm of f, relative to the function space at hand, is denoted by ε. For systems as (14.0.1), the previous results assert that the action variables are confined in a ball of radius $\mathbf{R}(\varepsilon)$ centered at the initial action during a time $\mathbf{T}(\varepsilon)$, provided that ε is smaller that some threshold $\mathbf{E}$. We say that $\mathbf{R}(\varepsilon)$ is the *confinement radius*, $\mathbf{T}(\varepsilon)$ is the stability time and $\mathbf{E}$ is the *applicability threshold*. The remarkable fact is that – h being given – the results depend only on the norm of f and not on its particular form.

Much attention has been paid in the literature in order to obtain good estimates for the quantities $\mathbf{R}(\varepsilon)$ and $\mathbf{T}(\varepsilon)$ in the different frameworks. As we shall see in the sequel, in the setting of Hölder perturbations of analytic integrable systems, the method we introduce in this part yields sharper estimates than those that are found in the literature

[1] Assuming that the unperturbed system is analytic is just a matter of simplification.

up to now. Before stating rigorously our results, however, it is useful to have an overview of the classical results on the effective stability of near-integrable Hamiltonian systems.

14.1 The classical results

Let us briefly describe the classical abstract results. In the 70's Nekhoroshev proved his seminal theorem [95], which asserts that for a steep real-analytic function h and for any real-analytic perturbation f with analytic extension to a complex domain $\mathcal{D}$, all solutions are stable at least over exponentially long time intervals. Namely, there exist positive *exponents* a, b and a positive threshold $\mathbf{E}$, depending only on h, such that if $|f|_{\mathcal{D}} \leq \mathbf{E}$, then any initial condition (I_0, θ_0) gives rise to a solution $\big(I(t), \theta(t)\big)$ which is defined at least for $|t| \leq \exp\big(c(1/\varepsilon)^a\big)$ and satisfies $\big|I(t) - I_0\big| \leq C\varepsilon^b$ in that range. Here $|f|_{\mathcal{D}}$ is the C^0 sup-norm on the domain $\mathcal{D}$ and c, C are positive constants which also depend only on h. With our notation, for these systems:

$$\mathbf{T}(\varepsilon) = \exp(c(1/\varepsilon)^a), \qquad \mathbf{R}(\varepsilon) = C\varepsilon^b, \tag{14.1.1}$$

while the expression of the threshold $\mathbf{E}$ is quite difficult to obtain explicitly[2], see [95]. Since the constants c and C are less significant than the exponents we will get rid of them in our subsequent description.

Nekhoroshev's proof is based on the construction of a partition (a "patchwork") of the phase space into zones of approximate resonances of different multiplicities, over which one can construct adapted normal forms. The global stability result necessitates a very delicate control of the size and disposition of the elements of the patchwork in order to produce a "dynamical confinement" preventing the orbits from fast motions along distances larger than the confinement radius (see below for a discussion).

In the convex case, as noticed in [64] and [20], a shrewd use of energy conservation leads to a much simpler and "physical" way to confine the orbits. This gave rise to two distinct series of works, originating in the articles of Lochak [83] - where the simultaneous approximation method was introduced - and Pöschel [104] - where the construction of Nekhoroshev's patchwork was made much easier - both relying on the convexity or quasi-convexity of the integrable Hamiltonian.

The simplicity of these methods made it possible to prove that the Nekhoroshev Theorem in the analytic case holds with

$$\mathbf{T}(\varepsilon) = \exp(c(1/\varepsilon)^{1/2n}), \qquad \mathbf{R}(\varepsilon) = C\varepsilon^{1/2n}, \tag{14.1.2}$$

if h is assumed to be quasi-convex (see [83, 85, 104]). Moreover, besides the global result, one can state local results for neighborhoods of resonant surfaces. For $m \in \{1, \dots, n-1\}$, consider a sublattice $\Lambda \in \mathbb{Z}^n_K := \{k \in \mathbb{Z}^n : |k|_1 \leq K\}$ of rank m and the resonant subset $\mathcal{M}_\Lambda := \{I \in \mathbb{R}^n \mid \nabla h(I) \in \Lambda^\perp\}$. Then, for all trajectories starting

[2] Thresholds have been studied more extensively in applications to celestial mechanics, see e.g. [97] or [15]

at a distance of order $\varepsilon^{1/2}$ of $\mathcal{M}_\Lambda$, one gets larger stability exponents, namely $a = b = 1/(2(n-m))$. Moreover, in the resonant block $\mathcal{B}_\Lambda$ (which is obtained by eliminating from $\mathcal{M}_\Lambda$ all the intersections with other resonant subsets $\mathcal{M}_{\Lambda'}$, with rank $\Lambda' = m+1$) one can even take $a = 1/(2(n-m)),\ b = 1/2$.

As alluded to above, long time stability does not require *a priori* the analyticity of the Hamiltonian at hand. For general Gevrey quasi-convex systems[3], the fast decay of the Fourier coefficients also yields exponentially long stability times. Namely, for β-Gevrey systems (where β is the Gevrey exponent) it is proved in [87] that

$$\mathbf{T}(\varepsilon) = \exp\left(c/\varepsilon^{1/(2n\beta)}\right), \qquad \mathbf{R}(\varepsilon) = C\varepsilon^{1/(2n\beta)}.$$

The proof is based on a direct construction of normal forms for Gevrey systems. This study was initiated by M. Herman for proving the optimality of the stability exponents by constructing explicit examples taking advantage of the flexibility of the Gevrey category, see below.

Soon after, finitely differentiable systems have been investigated in [30] using a *direct* implementation of Lochak's scheme in this setting, which yields the estimates

$$\mathbf{T}(\varepsilon) = c/\varepsilon^{(\ell-2)/(2n)} \qquad \mathbf{R}(\varepsilon) = C\varepsilon^{1/(2n)}$$

for quasi-convex C^ℓ systems with $\ell \geq 2$ and integer. On the other hand, the stability of C^ℓ systems, with ℓ an integer such that $\ell \geq \ell^* n + 1$ for some suitable $\ell^* \geq 1, \ell^* \in \mathbb{N}$, satisfying a property known as Diophantine-Morse condition[4], was investigated in [31], where the values

$$\mathbf{T}(\varepsilon) = c/\varepsilon^{\ell^*/[3(4(n+1))^n]} \qquad \mathbf{R}(\varepsilon) = C\varepsilon^{1/(4(n+1))^n}$$

were found.

The case $\ell = +\infty$ has been studied in [10], where the authors find that, in the case $h(I) = I^2/2$ and for fixed $b \in (0, 1/2)$, for any $M > 0$ there exists $C_M > 0$ such that

$$\mathbf{T}(\varepsilon) = \frac{C_M}{\varepsilon^M} \qquad \mathbf{R}(\varepsilon) = C_M\varepsilon^b\ .$$

The result is achieved by implementing an innovative *global* normal form in Pöschel's framework.

Finally, we also refer to the recent work [33] and references therein for much more information about stability in various functional classes.

[3] See [87] for the definition.

[4] The Diophantine-Morse property is a special case of the Diophantine-steep condition introduced in [99] which, in turn, is a prevalent condition on integrable systems that ensures long time stability once these are perturbed. All steep functions are Diophantine-steep.

14.2 Purpose of the work

The objective of this part is to make a systematic use of analytic smoothing methods to derive normal forms in a very simple way - whatever the regularity of the Hamiltonians at hand - from the usual analytic ones. This way we get maximal flexibility to adapt the different long-time stability proofs to a large class of function spaces. We will investigate here only the case of Hölder differentiable Hamiltonians, but our method extends to any steep functions belonging to any regularity class which admits an analytic smoothing. More precisely, the proposed strategy (see Section 17.3) allows us to prove, in a very simple way, the first Nekoroshev-type result of stability for Hölder steep Hamiltonians with presumed sharp exponents[5]. In this case one cannot expect to get more than polynomial stability times relative to the size ε of the perturbation [30]. In the course of the proof we need to adjust in a rather unusual way the size of the various parameters: ultraviolet cutoff and, in an essential way, the analyticity width, as a function of the size ε of the perturbation.

14.3 Main results

Let us fix the main definitions and assumptions. In the following, given $\nu \in \{1, \dots, \infty\}$, we denote by $|\cdot|_\nu$ the corresponding ℓ^ν-norm in $\mathbb{R}^n$ or $\mathbb{C}^n$. We denote by $B_\nu(I_0, R)$ the open ball centered at I_0 of radius R for the norm $|\cdot|_\nu$ in $\mathbb{R}^n$.

Consider a Hamiltonian of the form (14.0.1), where we assume, *for the sake of simplicity*, that the unperturbed part h is analytic[6] while only the perturbation f is Hölder, so:

$$h \in C^\omega(\mathbb{B}^R_{\rho_0}), \qquad f \in C^\ell(\mathbb{B}^R \times \mathbb{T}^n), \tag{14.3.1}$$

where $\mathbb{B}^R_{\rho_0}$ is the complex extension of analyticity width $\rho_0 \geq 1$ of $\mathbb{B}^R$, and $\ell \in (1, +\infty)$ (meaning that f is Hölder differentiable when ℓ is not an integer, see section 16 for a brief overview on this class of functions). The small parameter is

$$\varepsilon := |f|_{C^\ell(\mathbb{B}^R \times \mathbb{T}^n)}, \tag{14.3.2}$$

(see (16.1.2) for a definition of the Hölder norm). We denote by $\omega = \nabla h : \mathbb{R}^n \to \mathbb{R}^n$ the action-to-frequency map attached to h.

We will assume that the Hessian of h is uniformly bounded from above:

$$M := \sup_{I \in \mathbb{B}^R_{\rho_0}} \left\| D^2 h(I) \right\|_{op} < \infty, \tag{14.3.3}$$

where $\| \, \|_{op}$ stands for the operator norm induced by the Hermitian norm on $\mathbb{C}^n$.

[5] Sharpness has the same meaning as in [70], i.e. these are the best values of the exponents for $\mathbf{T}(\varepsilon)$ and $\mathbf{R}(\varepsilon)$ that one can obtain with these techniques.

[6] As we will see in the course of the proof, assuming that h is Hölder with large enough exponents would be enough, see Section 17.3.2

We will also assume that the Hamiltonian h is steep according to the following definition.

Definition 14.3.1 (Steepness)**.** Fix $\delta > 0$. A C^1 function $h : B_\infty(0, R+\delta) \to \mathbb{R}$ is *steep* with *steepness indices* $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1} \geq 1$ and *steepness coefficients* $C_1, \dots, C_{n-1}, \delta$ if:

1. $\inf_{B_\infty(0,R)} |\omega(I)|_2 > 0$;
2. for any $I \in B_\infty(0, R)$ and any m-dimensional subspace Γ orthogonal to $\omega(I)$, with $1 \leq m < n$:

$$\max_{0\leq\eta\leq\xi} \min_{u\in\Gamma, |u|_2=\eta} \left|\pi_\Gamma \omega(I+u)\right|_2 > C_m \xi^{\boldsymbol{\alpha}_m}, \quad \forall \xi \in (0, \delta], \tag{14.3.4}$$

   where π_Γ stands for the orthogonal projection on Γ.

Remark 14.3.1. Note that a uniformly strictly convex function is steep with steepness indices equal to 1.

Remark 14.3.2. The steepness condition is generic in the space of jets of sufficiently regular functions (see [94] for the general discussion and [111], [13] for sufficient conditions for steepness in the space of jets of order four and five respectively).

Our main theorem is the following.

Theorem 14.3.1 (Stability estimates in the steep case)**.** *Consider a near-integrable Hamiltonian system* (14.0.1) *satisfying* (14.3.1) *and assume* $\ell \geq n+1$ [7]. *Suppose that h is steep in* $B_\infty(0, R)$ *with steepness indices* $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1})$ *and set:*

$$\mathtt{a} := \frac{\ell - 1}{2n\boldsymbol{\alpha}_1 \times \cdots \times \boldsymbol{\alpha}_{n-2}} + \frac{1}{2} \quad , \qquad \mathtt{b} := \frac{1}{2n\boldsymbol{\alpha}_1 \times \cdots \times \boldsymbol{\alpha}_{n-1}} \, .$$

Then, there exist positive constants $\mathbf{E} = \mathbf{E}(n, \ell, \boldsymbol{\alpha})$, $\mathtt{C}''_{\mathbf{I}} := \mathtt{C}''_{\mathbf{I}}(n, \ell, \boldsymbol{\alpha})$, $\mathtt{C}''_{\mathbf{T}} := \mathtt{C}''_{\mathbf{T}}(n, \ell, \boldsymbol{\alpha})$ *such that, for* $\varepsilon \leq \mathbf{E}$*, the radius and time of confinement relative to any initial condition in the set* $B_\infty(0, R/4)$ *satisfy:*

$$\mathbf{R}(\varepsilon) \leq \mathtt{C}''_{\mathbf{I}} \varepsilon^{\mathtt{b}} \quad , \qquad \mathbf{T}(\varepsilon) \leq \mathtt{C}''_{\mathbf{T}} \frac{1}{|\ln \varepsilon|^{\ell-1} \, \varepsilon^{\mathtt{a}}} \, . \tag{14.3.5}$$

Remark 14.3.3.
• The presence of the logarithm in (14.3.5) comes from the fact that in our method we have some freedom to fix the analyticity width depending on ε, in contrast with the classical analytic setting. We send the reader to Remark 18.2.1, where this comment is contextualized, the dependence of the analyticity width in ε is made precise and a qualitative justification is given.
• If we set $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1} = 1$ (i.e. the convex case) we obtain better estimates than

[7] Actually one could probably get $\ell \gtrsim n/2$ by making use of Paley-Littlewood theory.

in [30].
• Our proof relies on the geometric construction of the geography of resonances introduced in [70], which is appropriate only for Hamiltonians in $n \geq 3$ degrees of freedom. Here too we shall restrict to this setting, the 2 d.o.f. isoenergetic non-degenerate case being easily managed through KAM theory. A specific construction should be implemented to treat the peculiarity of the isoenergetic degenerate 2 d.o.f. case. This study is in progress in a forthcoming work.

14.4 Prospects

The sharpness of the exponents in Theorem 14.3.1 should be proved in the same way as in the case of convex system. The first attempt to tackle this problem led to work in the Gevrey category instead of the analytic one and construct examples with unstable orbits, which experience a drift in action of the same order as the confinement radius within a time of the same order as the stability time, see [87]. It has then be realized that the initial conjecture in quasi-convex analytic systems ($a \sim 1/2n$, see [49] and Lochak [83]) was in fact incorrect: as proved in [35] using a purely topological argument together with the previous remark on the *local* exponents near simple resonances, one can choose $a = 1/(2(n-1))$ as a global stability exponent for $\mathbf{T}(\varepsilon)$. This result was improved soon after with $a \sim 1/(2(n-2))$ (see [120]). The construction of unstable system proving the optimality of these latter exponents was achieved in [87], [84], [120]. A remarkable fact is that the unstable mechanism introduced by Arnold in the 60's, with its subsequent improvements, is exactly what is needed to produce the unstable examples in the quasi-convex case.

As for the steep case, a careful construction of the geography of resonances leads with strong evidence to the conjecture that the exponents $a = 1/(2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-2})$ and $b = 1/(2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-1})$ are sharp (see ref. [70]). The question of constructing explicit examples with unstable orbits proving this sharpness is still open nowadays and is maybe the last challenging problem in the general long time stability theory, probably relying on new Arnold diffusion ideas.

The part is organized as follows: in the next section we give a short overview of the classical methods with particular attention on the geometry of resonant blocks, on which the present work strongly relies. Next we define the functional setting. In Section 17 we introduce the analytic smoothing appropriately adapted to our problem. Finally Section 18 is devoted to the study of the steep case.

Chapter 15

General setting and classical methods

15.1 Resonances and the steepness condition.

Consider a Hamiltonian system of the form (14.0.1) defined on $O \times \mathbb{T}^n$, where O is an open subset of $\mathbb{R}^n$. The main feature underlying Hamiltonian perturbation theory is that one can modify the form of the perturbation f by composing H with properly chosen local Hamiltonian diffeomorphisms, in order to remove a large number of "nonessential harmonics". The result of this process - a local normal form - strongly depends on the location of the domain of the normalizing diffeomorphism w.r.t the resonances of the unperturbed part h, and enables one to discriminate between fast drift and extremely slow drift directions in the action space, according to this location.

Let us first make this idea more precise. Given an integer lattice $\Lambda \subset \mathbb{R}^n$ of dimension $m \in \{1, \dots, n-1\}$ – a *resonance lattice* – one associates with Λ the resonance vector subspace $\Lambda^\perp \subset \mathbb{R}^n$ in the frequency space $\mathbb{R}^n$, together with the corresponding resonance subset in the action space previously introduced

$$\mathcal{M}_\Lambda := \omega^{-1}(\Lambda^\perp) = \{I \in O \mid \omega(I) \in \Lambda^\perp\},$$

where $\omega = \nabla h$ is the frequency map. The dimension m of Λ is said to be the multiplicity of the resonance $\mathcal{M}_\Lambda$. Of course, given a resonance module $\Lambda' \supset \Lambda$ with $\dim \Lambda' > \dim \Lambda$, the resonance $\mathcal{M}_{\Lambda'}$ is contained in $\mathcal{M}_\Lambda$, so that a resonance subset contains in general infinitely many resonances of higher multiplicity. The complement $\mathcal{M}_0 \subset O$ of the union of all resonance subsets is the *non-resonant subset*. In general, a resonance subset $\mathcal{M}_\Lambda$ has no particular structure, however, one can think of $\mathcal{M}_\Lambda$ as a submanifold of $\mathbb{R}^n$ of the same dimension as $\Lambda^\perp$ (with perharps singular loci).

As a rule, when ε is small enough, for a small enough ε-depending neighborhood W_Λ of the parts of the resonance subset $\mathcal{M}_\Lambda$ *located far enough from resonances of* *higher multiplicity*[1], one can iteratively construct a symplectic diffeomorphism Ψ_Λ, whose image contains $W_\Lambda \times \mathbb{T}^n$, such that the pull-back $H_\Lambda = H \circ \Psi_\Lambda$ takes the following form

$$H_\Lambda = h + N_\Lambda + R_\Lambda. \tag{15.1.1}$$

Here R_Λ is a remainder whose C^2 norm is (very) small[2] with respect to ε and the *resonant part* N_Λ contains only harmonics belonging to Λ, that is:

$$N_\Lambda(I,\theta) = \sum_{k\in\Lambda,\, |k|_1 \le K(\varepsilon)} a_k(I)\, e^{ik\cdot\theta},$$

where $K(\varepsilon)$ is an ultraviolet cutoff which has to be properly chosen[3]. Both terms N_Λ and R_Λ of course depend on ε. A subset W_Λ for which such a normal form is proved to exist will be called a *normal form neighborhood associated with* Λ, with multiplicity $\dim \Lambda$. One proves that the space of actions can be covered by such neighborhoods, and in Section 18.1, we will construct finer covers by subsets of those, named *resonant blocks* (and denoted by D_Λ in the aforementioned section).

The iterative process to construct the normalizing diffeomorphism involves the control of small denominators which appear during the resolution of the so-called homological equation, and which depend on the location of the normalization domain with respect to the resonances (see for instance [104]). This can be seen as a drawback of the method which could be greatly simplified by an idea due to Lochak (see below), however the general method presented here give precise dynamical informations which would not be reachable otherwise.

The Hamilton equations generated by (15.1.1) yield the following form for the evolution of the action variables:

$$\begin{aligned} I(t) - I(0) &= \int_0^t \partial_\theta N_\Lambda\,(I(s),\theta(s)) + \partial_\theta R_\Lambda\,(I(s),\theta(s))\; ds \\ &= \sum_{k\in\Lambda,\, |k|_1\le K(\varepsilon)} k\cdot\Big(\int_0^t i\, a_k(I(s))\, e^{ik\cdot\theta(s)}\, ds\Big) + \mathcal{R}(t). \end{aligned} \tag{15.1.2}$$

The variation of I is therefore the sum of the main part

$$\mathcal{D}(t) := \sum_{k\in\Lambda,\, |k|_1\le K(\varepsilon)} k\cdot \mathcal{N}^{(k)}(t), \qquad \mathcal{N}^{(k)}(t) = \int_0^t i\, a_k(I(s))\, e^{ik\cdot\theta(s)}\, ds, \tag{15.1.3}$$

and the very small remainder term $\mathcal{R}(t)$.

To simplify the presentation in the following, we will *forget about the angles* and consider only the action part of the solutions of our system (which is legitimized by the fact that the angles play no role in the various estimates).

[1] In fact, only a finite ε-depending subset (related to the cutoff $K(\varepsilon)$ introduced below) of these resonances has to be taken into account.

[2] The smallness depends on the regularity of the system.

[3] This choice is indeed a main issue in the theory.

The whole theory relies firstly on the obvious fact that the main drift term $\mathcal{D}(t)$ in (15.1.3) belongs to the vector space Vect Λ spanned by Λ (which is often called *"plane" of fast drift*), and secondly on the smallness of the remainder term $\mathcal{R}$. A solution starting from some initial condition $I(0) \in W_\Lambda$ will therefore remain very close to the fast drift space

$$I(0) + \mathrm{Vect}\,\Lambda$$

during a very long time – governed by the smallness of $\mathcal{R}$ – *as long as it is contained inside the neighborhood* W_Λ. This makes it necessary to understand first the intersections of the fast drift planes $I + \mathrm{Vect}\,\Lambda$ and the neighborhoods W_Λ to which they are attached.

As an extreme example, let us consider the Hamiltonian

$$h(I) = \frac{1}{2}(I_1^2 - I_2^2)$$

on $\mathbb{A}^2$, with (invertible) frequency map $\omega(I_1, I_2) = (I_1, -I_2)$. We focus on the resonance module $\Lambda = \mathbb{Z}(1, -1)$, so that $\Lambda^\perp = \mathbb{R}(1, 1)$ and Vect $\Lambda = \mathcal{M}_\Lambda$. Hence, given an initial action $I(0) \in \mathcal{M}_\Lambda$, the entire fast drift affine subspace $I(0) + \mathrm{Vect}\,\Lambda$ coincides with $\mathcal{M}_\Lambda$, so that nothing prevents the fast drift to take place during the whole motion provided the perturbation is well-chosen: the resonance $\mathcal{M}_\Lambda$ is called a *superconductivity channel*. No long time stability result can be expected in this case: indeed, when $f(I, \theta) = \sin(\theta_1 - \theta_2)$, the initial condition $I = 0$, $\theta = 0$ yields the fast evolution $(I_1(t), I_2(t)) = (-\varepsilon t, \varepsilon t)$ for the action variables [4].

In constrast with the previous example, for the Hamiltonian

$$H(I, \theta) = \frac{1}{2}|I|_2^2 + \varepsilon f(\theta)$$

on $\mathbb{A}^n$, for any $\Lambda \subset \mathbb{Z}^n_K$, the the resonant set $\mathcal{M}_\Lambda$ coincides with $\Lambda^\perp$, so that the affine planes of fast drift are always orthogonal to $\mathcal{M}_\Lambda$. In this case a fast drift - if it happens - makes the orbits move away from the resonance in a very short time.

These extreme examples illustrate the role of the Nekhoroshev condition: steepness is an intermediate quantitative property, which prevents from the existence of the superconductivity channels by ensuring a certain amount of transversality between the fast drift planes and the corresponding resonances in action. Starting from an action $I = I(0)$ located at some resonance $\mathcal{M}_\Lambda$, so that its associated frequency $\omega(I)$ is orthogonal to $\Gamma := \mathrm{Vect}\,\Lambda$, the condition

$$\max_{0 \le \eta \le \xi} \min_{u \in \Gamma, |u|_2 = \eta} \left|\pi_\Gamma \omega(I + u)\right|_2 > C_m \xi^{\alpha_m}, \quad \forall \xi \in (0, \delta], \tag{15.1.4}$$

(where π_Γ stands for the orthogonal projection on Γ) imposes that a drift of length ξ starting from I and occuring along the fast drift plane $I + \Gamma$ makes the projection $\pi_\Gamma(\omega)$ change by an amount of $C_m \xi^{\alpha_m}$ during the way.

[4] Here a proper choice of the initial angles is needed.

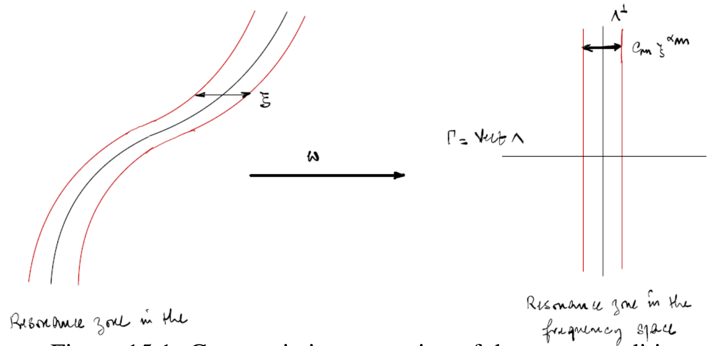

Figure 15.1: Geometric interpretation of the steep condition

This admits an easy geometric interpretation (see Figure 1). Assume $\dim \Lambda = m$ and consider the vector space Γ spanned by Λ, together with its orthogonal space $\Lambda^\perp$ - of dimension $n - m$. Then one can define a family of tubular neighborhoods of $\Lambda^\perp$ of width $\delta > 0$ by

$$\mathbf{T}_\delta(\Lambda^\perp) = \{\varpi \in \mathbb{R}^n \mid \pi_\Gamma(\varpi) < \delta\}, \quad \delta > 0 . \tag{15.1.5}$$

Each such neighborhood gives rise to a neighborhood of the resonance $\mathcal{M}_\Lambda$ in action, namely:

$$\mathbf{W}_\delta(\mathcal{M}_\Lambda) = \omega^{-1}\big(\mathbf{T}_\delta(\Lambda^\perp)\big). \tag{15.1.6}$$

Therefore, condition (15.1.4) just says that *any* orbit starting from I and drifting to a distance ξ from I along the plane of fast drift Γ must exit the neighborhood $\mathbf{W}_\delta(\mathcal{M}_\Lambda)$ with $\delta = C_m \xi^{\alpha_m}$.

Note finally that given disjoint subsets $\mathbf{T}$, $\mathbf{T}'$ of tubular neighborhoods of the form (15.1.5), the associated neighborhoods $\omega^{-1}(\mathbf{T})$ and $\omega^{-1}(\mathbf{T}')$ are disjoint too, whatever the geometric assumptions on the frequency map ω.

15.2 Nekhoroshev's hierarchy.

This section is inspired by Nekhoroshev's ideas as presented in the very nice paper [70]. We also refer to [69] for further details and to [99] for a different approach. Nekhoroshev's strategy to prove long-time stability results for perturbations of steep Hamiltonians is based on the previous description of resonant neighborhoods, and relies on the following key observation.

Given ε small enough, there exist $T(\varepsilon)$, $R(\varepsilon)$ and a covering of the action space O by resonant "blocks" $(\mathcal{B}_{m,p})_{0\le p\le p_m}$, for $0 \le m \le n-1$, and $m, p, p_m \in \mathbb{N}$, which satisfy the following properties:

1. *$T(\varepsilon) \to +\infty$ and $R(\varepsilon) \to 0$ when $\varepsilon \to 0$;*

2. *each block $\mathcal{B}_{m,p}$ is contained in a resonant neighborhood of multiplicity m and admits an enlargement $\widehat{\mathcal{B}}_{m,p} \supset \mathcal{B}_{m,p}$ contained in the same resonant neighborhood;*

3. *any solution starting from an initial condition in $\mathcal{B}_{m,p}$ either stays inside $\widehat{\mathcal{B}}_{m,p}$ for $0 \le t \le T(\varepsilon)$ or admits a first exit time t_1 such that $I(t_1)$ belongs to a block $\mathcal{B}_{m',p'}$ with $m' < m$;*

4. *for any initial condition $I(0)$ inside a block $\mathcal{B}_{m,p}$ and for any interval $\mathcal{I}$ such that $I(t) \in \widehat{\mathcal{B}}_{m,p}$ for all $t \in \mathcal{I}$, then*

$$|I(t) - I(0)|_2 < R(\varepsilon), \quad \forall t \in \mathcal{I}.$$

We say that m is the multiplicity of the block $\mathcal{B}_{m,p}$. Taking the previous observation for granted, the stability of the action variable over a timescale $T(\varepsilon)$ is easy to prove by finite induction. Given an initial condition $I(0)$ located in some block $\mathcal{B}_{m_0,p_0}$, either $I(t) \in \widehat{\mathcal{B}}_{m_0,p_0}$ for $0 \le t \le T(\varepsilon)$, or there is a t_1 such that $I(t) \in \widehat{\mathcal{B}}_{m_0,p_0}$ for $0 \le t < t_1$ and $I(t_1)$ belongs to a block $\mathcal{B}_{m_1,p_1}$ with $m_1 < m_0$. Consequently, there is a finite sequence $(m_0, p_0), \ldots, (m_j, p_j)$ such that $m_0 > m_1 > \cdots > m_j$ (with maybe $m_j = 0$) and a finite sequence of times $t_0 = 0 < t_1 < \cdots < t_p = T(\varepsilon)$ such that for $0 \le i < j$:

$$I(t) \in \widehat{\mathcal{B}}_{(m_i,p_i)}, \quad \forall t \in [t_i, t_{i+1}].$$

In words, any orbits crosses a finite number of enlarged blocks during the interval $[0, T(\varepsilon)]$ and get trapped inside the last one. To conclude, one just has to use property (4), which proves that the distance between $I(0)$ and $I(t)$ is at most $nR(\varepsilon)$ for $t \in [0, T(\varepsilon)]$.

One should be aware that the covering by the blocks is *not* a partition of O: two distinct blocks may have a nonempty intersection. However, one can *choose* the blocks visited by the orbits according to a hierarchical order, in such a way that their multiplicity decreases as t increases [5]. We say that a covering of O by blocks satisfying the previous properties is a Nekhoroshev patchwork.

15.3 Construction of Nekhoroshev patchworks.

Let us now describe how the blocks are constructed so as to possess their covering and confinement properties[6].

Given $\varepsilon > 0$, we first fix an ultraviolet cutoff $K(\varepsilon)$ and consider only the set $\mathbf{M}_\varepsilon$ of resonance modules which are spanned by vectors of length smaller than $K(\varepsilon)$. Given a

[5] This raises the question of the existence of local finite time Lyapunov functions on the phase space, a still unclear issue.

[6] A source of inspiration for nowadays governments.

resonant module $\Lambda \in \mathbf{M}_\varepsilon$ of multiplicity m, we start with the resonant zone of "width" δ_Λ

$$Z_\Lambda := W_{\delta_\Lambda}(\mathcal{M}_\Lambda) = \omega^{-1}\left\{\varpi \in \mathbb{R}^n \mid \left|\pi_\Gamma(\varpi)\right|_2 < \delta_\Lambda\right\},$$

where δ_Λ has to be properly chosen as a function of ε and the various geometric invariants of the module (see section 18). We then define the (ε-dependent) resonant zone $\mathcal{Z}_m$ of multiplicity m as

$$Z_m = \bigcup_{\Lambda \in \mathbf{M}_\varepsilon,\, \dim \Lambda = m} Z_\Lambda.$$

Given $\Lambda \in \mathbf{M}_\varepsilon$, $\dim \Lambda = m$, the block attached to Λ is obtained by removing from Z_Λ its intersection with the complete resonant zone of multiplicity $m + 1$:

$$\mathcal{B}_\Lambda = Z_\Lambda \setminus Z_{m+1}.$$

The blocks $\mathcal{B}_{m,p}$ are the connected components of Z_m. With no great loss of generality, one can think of (the closure of) a block as a submanifold with boundary and corners – even if it is not necessary.

The following figure shows the construction of the blocks in the case $n = 3$ (and in a transverse section). The resonance zone of multiplicity 2 if the disjoint union of the blue blocks, the resonance zone of multiplicity 1 is the union on the strips with red boundaries, while the 0-multiplicity zone is the complement of the 1-multiplicity zone.

In any case, the blocks satisfy two main properties.

– The closures of two different blocks can intersect only when their multiplicities are distinct.

This comes from a very careful choice of the widths of the various resonance zones (see [70] and Section 18), which in fact ensures a more stringent (and crucial) property: the enlargement of a block contained in some $\mathcal{B}_\Lambda$ cannot intersect *any other block* contained in the zone $\mathcal{B}_\Lambda$, neither *any other neighborhood* $\mathcal{M}_{\Lambda'}$ *with* $\dim \Lambda' = \dim \Lambda$ (see below for precisions on the construction of the enlargement).

We state the second property in the spirit of Conley's isolating blocks theory.

– The frontier $\partial\mathcal{B}_{m,p}$ of $\mathcal{B}_{m,p}$ is the union of two subsets

$$\partial\mathcal{B}_{m,p} = \partial^+\mathcal{B}_{m,p} + \partial^-\mathcal{B}_{m,p}$$

where $\partial^+\mathcal{B}_{m,p}$ (resp. $\partial^-\mathcal{B}_{m,p}$) is contained in blocks $\mathcal{B}_{m',p'}$ with $m' > m$ (resp. $m' < m$).

This raises new questions which could be the starting point of a better understanding of the relations between diffusion along invariant subsets and long-time stability theory. Indeed, given a block $\mathcal{B}_{m,p}$, a description of the (generic) features of the Hamiltonian vector field X_{H_ε} at the frontier $\partial\mathcal{B}_{m,p}$ has never been done. In particular, nothing is

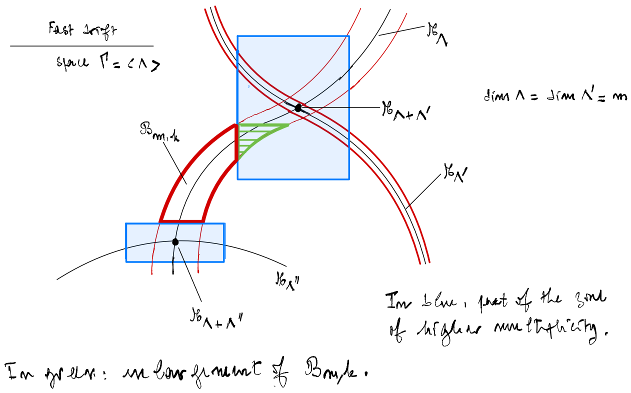

Figure 15.2: Construction of the resonant blocks

known on the locus where X_{H_ε} "enters the block" and the locus where X_{H_ε} "exits the block". These two subsets are crucial for the understanding of the homology of the invariant sets contained into the blocks, following Conley's theory, and could provide one with a new tool for constructing diffusing orbits in the steep setting.

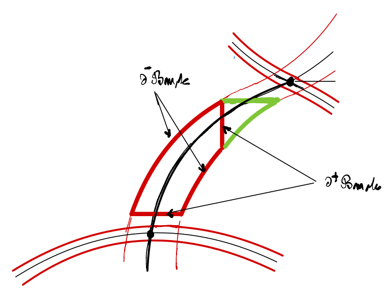

Figure 15.3: Interpretation of the resonant blocks in the light of Conley's theory

Going back to the construction of Nekhoroshev's patchwork, we have to make precise the process conducting to the enlargement of a block and its stability property. Here we will again make a crucial use of the fact that an orbit starting from an initial condition $I := I(0)$ located in $\mathcal{B}_{m,p}$ will remain extremely close to the fast drift space $I + \mathrm{Vect}\,\Lambda$ for $0 \leq t \leq T(\varepsilon)$, as long as it stays inside the resonant neighborhood $\mathcal{M}_\Lambda$ and far enough to the higher multiplicity resonance zones. Hence, to enlarge the block $\mathcal{B}_{m,k}$, we just have to add to it the collection of all the parts of the disks centered at points $I \in \mathcal{B}_{m,p}$ which are contained in the intersection of the fast drift spaces $I + \mathrm{Vect}\,\Lambda$ with the resonant neighborhood $\mathcal{M}_\Lambda$ (the resulting added subset is the green part in the previous two figures). We have in fact to add a very small neighborhood of these union of disks, in order to prevent the solutions to exit the extended block under the influence of the remainder part $\mathcal{R}$ of the dynamics during the time $T(\varepsilon)$, but this would not change

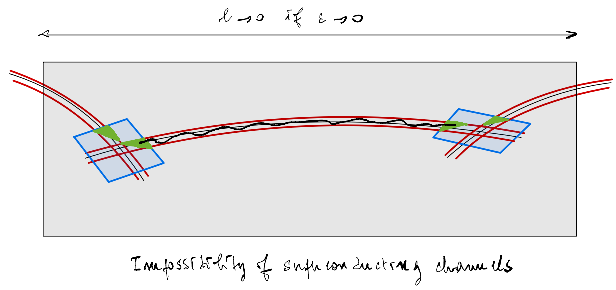

Figure 15.4: The Steepness property prevents the existence of superconductivity channels by ensuring a contact of finite order between the resonant manifold and the plane of fast drift. Here in the figure, ℓ is the size of the resonant zone (see Section 18.1)

our description significantly. Finally, one has to make sure that the extension will not intersect any other block of the same neighborhood $\mathcal{B}_\Lambda$ or any other resonance neighborhood, which can be done by a careful tuning of the width of the zone (see Section 18).

This concludes our description of Nekhoroshev's method.

Chapter 16

Functional setting

For $n \geq 1$, we denote the standard n-dimensional torus by $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and the standard $2n$-dimensional annulus by $\mathbb{A}^n = \mathbb{R}^n \times \mathbb{T}^n$.

16.1 Hölder differentiable functions.

Given an integer $q \geq 0$ and an open subset D of $\mathbb{R}^n$, we denote by $C^q(D)$ the set of q-times continuously differentiable maps $f : D \to \mathbb{R}$ ($C^0(D)$ being the set of continuous functions on D). We identify $C^q(\mathbb{T}^n)$ with the subset of $C^q(\mathbb{R}^n)$ formed by the functions that are $2\pi\mathbb{Z}^n$-periodic and $C^q(D \times \mathbb{T}^n)$ with the subset of $C^q(D \times \mathbb{R}^n)$ formed by the functions which are $2\pi\mathbb{Z}^n$-periodic with respect to their last n variables.

We use the conventional notation for partial derivatives: given $f \in C^q(D)$ and $\alpha \in \mathbb{N}^n$, we set for $x \in D$:

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x),$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We denote by $C_b^q(D)$ the set of $f \in C^q(D)$ such that

$$\|f\|_{C^q(D)} := \sup_{|\alpha| \leq q} \sup_{x \in D} |\partial^\alpha f(x)| < +\infty, \tag{16.1.1}$$

so that $\left(C_b^q(D), \|\cdot\|_{C^q(D)}\right)$ is a Banach space with multiplicative norm[1]. It is understood that, for a function defined on a compex domain D, the $\|\cdot\|_{C^0(D)}$ is the usual sup-norm.

If $\ell > 0$ is a non-integer real number, we write $q := \lfloor \ell \rfloor$ for its integer part and $\mu = \ell - q \in (0, 1)$ for its fractional part. Given a non-negative integer q and $\mu \in (0, 1)$,

[1] That is, satisfying an inequality of the form $|fg| \leq C|f||g|$ for a suitable constant C.

we denote by $C_b^{q,\mu}(D)$ the space formed by those functions $f \in C^q(D)$ such that

$$|f|_{C^{q,\mu}(D)} := \|f\|_{C^q(D)} + \sup_{\alpha\in\mathbb{N}^n : |\alpha|=q} \sup_{\substack{x,y\in D: \\ 0<|x-y|<1}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\mu} < +\infty. \tag{16.1.2}$$

It is well-known that $\left(C_b^{q,\mu}(D), |\cdot|_{C^{q,\mu}(D)}\right)$ is also a Banach space with multiplicative norm. Functions belonging to these spaces are called Hölder-differentiable functions.

Given a non-integer real number $\ell > 0$, together with its integer part $q := \lfloor \ell \rfloor$ and its fractional part $\mu = \ell - q \in (0,1)$, we also write $C_b^\ell(D)$ instead of $C_b^{q,\mu}(D)$ and $|\cdot|_{C^\ell(D)}$ instead of $|\cdot|_{C^{q,\mu}(D)}$. Clearly $C_b^\ell(D) \subset C_b^{\ell'}(D)$ when $\ell \geq \ell'$ and if $f \in C_b^\ell(D)$

$$|f|_{C^{\ell'}(D)} \leq |f|_{C^\ell(D)}. \tag{16.1.3}$$

16.2 Domains and their complex extensions.

Let us define the complex n-dimensional torus $\mathbb{T}^n_{\mathbb{C}}$ and the complex $2n$-dimensional annulus $\mathbb{A}^n_{\mathbb{C}}$ as

$$\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n / 2\pi\mathbb{Z}^n \qquad \text{and} \qquad \mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n \times \mathbb{T}^n_{\mathbb{C}}. \tag{16.2.1}$$

We use angle coordinates θ on $\mathbb{T}^n_{\mathbb{C}}$ (with the usual abuse $\theta \in \mathbb{C}^n$ when there is no ambiguity) and action-angle coordinates (I, θ) on $\mathbb{A}^n_{\mathbb{C}}$. We see $\mathbb{T}^n_{\mathbb{C}}$ as a real n-dimensional vector bundle over $\mathbb{T}^n$. Consequently, we write

$$|\theta| := \max_j \left(\left|\operatorname{Im}\theta_j\right|\right), \qquad |I| := \max_j \left|I_j\right|, \qquad |(I,\theta)| = \max\left(|I|, |\theta|\right). \tag{16.2.2}$$

For integer vectors $k \in \mathbb{Z}^n$, we use the "dual" ℓ^1-norm, which we write $|k|$ only when there is no risk of confusion.

We need to introduce specific domains in $\mathbb{A}^n_{\mathbb{C}}$. First, given $r > 0$, for a domain $D \subset \mathbb{R}^n$, we set

$$D_r := \left\{ z \in \mathbb{C}^n \,:\, \exists z^* \in D \,:\, |z - z^*|_2 < r \right\}. \tag{16.2.3}$$

As for the torus, given $s > 0$, we introduce the global complex neighborhood

$$\mathbb{T}^n_s := \left\{ \theta \in \mathbb{T}^n_{\mathbb{C}} \;:\; |\theta| < s \right\}. \tag{16.2.4}$$

We will essentially deal with complex domains of the form

$$\mathcal{D}_{r,s} := D_r \times \mathbb{T}^n_s \subset \mathbb{A}^n_{\mathbb{C}}. \tag{16.2.5}$$

We finally write $D_r^{\mathbb{R}}$ and $\mathcal{D}_{r,s}^{\mathbb{R}}$ for the projections of D_r and $\mathcal{D}_{r,s}$ on $\mathbb{R}^n$ and $\mathbb{A}^n$ respectively.

16.3 Analytic functions and norms.

If g is a bounded holomorphic function defined on $\mathbb{T}^n_s$, D_r or $\mathcal{D}_{r,s}$ we denote the corresponding classical sup-norms by

$$|g|_s = \sup_{\theta\in\mathbb{T}^n_s} |g(\theta)|, \quad |g|_r = \sup_{I\in D_r} |g(I)|, \quad |g|_{r,s} = \sup_{(I,\theta)\in\mathcal{D}_{r,s}} |g(I,\theta)|. \tag{16.3.1}$$

Fix a bounded holomorphic function $g : \mathcal{D}_{r,s+2\sigma} \to \mathbb{C}$, where $\sigma > 0$, and let $g(I,\theta) = \sum_{k\in\mathbb{Z}^n} \hat{g}_k(I)e^{\mathrm{i}\,k\cdot\theta}$ be its Fourier expansion, where $k\cdot\theta = k_1\theta_1 + \cdots + k_n\theta_n$. We then introduce the *weighted Fourier norm*

$$||g||_{r,s} := \sup_{I\in D_r} \sum_{k\in\mathbb{Z}^n} \left|\hat{g}_k(I)\right| e^{|k|s}, \tag{16.3.2}$$

which is finite and satisfies

$$|g|_{r,s} \le ||g||_{r,s} \le \coth^{\mathrm{n}}\sigma\, |g|_{r,s+\sigma}. \tag{16.3.3}$$

We denote by $\mathcal{A}_{r,s}$ the space of holomorphic functions on $\mathcal{D}_{r,s}$ with finite Fourier norm. Endowed with this norm, $\mathcal{A}_{r,s}$ is a Banach algebra.

Finally, the norm of a vector valued function will be the maximum of the norms of its components.

Chapter 17

Analytic smoothing

We state in this section the key ingredient of the present work. We first recall the analytic smoothing method as developed by Jackson-Moser-Zehnder for Hölder functions of $\mathbb{R}^n$: given a Hölder function $f \in C^\ell(\mathbb{R}^n)$ and a positive number $s \le 1$, this yields an analytic function on the complex neighborhood $\mathbb{R}^n_s$ whose restriction to $\mathbb{R}^n$ is close to f in the C^k topology, for $1 \le k \le \ell$.

We then adapt their method to our specific setting of functions defined on $\mathbb{A}^n$ (see Section 17.2) and, in addition, we derive the new estimate (17.3.2) for the weighted Fourier norm of the smoothed function.

17.1 Analytic smoothing in $\mathbb{R}^n$

We recall here the result by Jackson, Moser and Zehnder, following the presentation by [45] and [110].

Proposition 17.1.1 (Jackson-Moser-Zehnder)**.** *Fix an integer $n \ge 1$, a real number $\ell > 0$ and let $f \in C^\ell_b(\mathbb{R}^n)$. Then there is a constant $C_{\mathtt{J}} = C_{\mathtt{J}}(\ell, n)$ such that for every $0 < s \le 1$ there exists a function $\mathtt{f}_s$, analytic on $\mathbb{R}^n_s$, which satisfies*

$$\left| \partial^\alpha \mathtt{f}_s(x) - \sum_{\beta \in \mathbb{N}^n : |\beta| \le \lfloor \ell \rfloor - |\alpha|} \partial^{\alpha+\beta} f(\operatorname{Re} x) \frac{(\operatorname{Im} x)^\beta}{\beta!} \right| \le C_{\mathtt{J}}\, s^{\ell - |\alpha|} |f|_{C^\ell(\mathbb{R}^n)}, \quad \forall x \in \mathbb{R}^n_s, \tag{17.1.1}$$

for all multi-integer $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le \lfloor \ell \rfloor$. More precisely, given any even C^∞ function Φ with compact support in $\mathbb{R}^n$ and setting

$$K(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Phi(x) e^{\mathrm{i} x \cdot \xi} dx, \quad \xi \in \mathbb{R}^n_s, \tag{17.1.2}$$

the function

$$\mathtt{f}_s(x) := \int_{\mathbb{R}^n} K\left(\frac{x}{s} - \xi\right) f(s\xi)\, d\xi\,, \tag{17.1.3}$$

satisfies the previous requirements (where the constant $C_{\mathrm{J}}(\ell, n)$ depends on the choice of Φ).

Observe that $\mathtt{f}_s$ takes real values when its argument is in $\mathbb{R}^n$.

17.2 Analytic smoothing in $\mathbb{A}^n$.

In the following, the Hölder regularity ℓ is assumed to satisfy $\lfloor \ell \rfloor \geq n+1$ as in the hypotheses of Theorem 14.3.1.
We now specialize the previous result to our setting and give a more detailed description of the method in the case of functions of $\mathbb{A}^n$. In that case, the analytic smoothing is a truncation of the Fourier series of the initial Hölder function with suitably modified Fourier coefficients (the so-called Jackson polynomials). Our main concern here is to derive an estimate on the weighted Fourier norm of an s-smoothed C^ℓ function over a complex strip of width s.

To make the whole presentation more explicit *and take the anisotropy of the weighted Fourier norm into account*, we first consider functions defined on $\mathbb{R}^n$ and $\mathbb{T}^n$ separately. This then yields a statement for functions of $\mathbb{A}^n$.

• *The non-periodic case*. Fix an even function $\Phi : \mathbb{R}^n \to [0, 1]$, of class C^∞, with support in the ball $\overline{B}_2(0, 1)$ and let $K : \mathbb{C}^n \to \mathbb{C}$ be its Fourier-Laplace transform:

$$K(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Phi(\eta) e^{-i\eta \cdot y} d\eta. \tag{17.2.1}$$

Since Φ is compactly supported, then K is an entire function . Moreover its restriction to $\mathbb{R}^n$ is in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ since Φ is, and this is also the case for the translates $y \mapsto K(y - z)$ for $y \in \mathbb{R}^n$ and fixed $z \in \mathbb{C}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^ℓ function with $\lfloor \ell \rfloor \geq n+1$, with compact support contained in the ball $\overline{B}_\infty(0, R_0)$ for some $R_0 > 0$. Given $s \in]0, 1]$, set for $x \in \mathbb{R}^n$:

$$\mathbf{f}_s(x) = \frac{1}{s^n} \int_{\mathbb{R}^n} K\Big(\frac{x-y}{s}\Big) f(y) dy = \int_{\mathbb{R}^n} K\Big(\frac{x}{s} - y\Big) f(sy) dy = \int_{\mathbb{R}^n} K(y) f(x - sy) dy. \tag{17.2.2}$$

By Fourier reciprocity:

$$\mathbf{f}_s(x) = \int_{\mathbb{R}^n} \Phi(\eta) \widehat{f(x - sy)}(\eta) d\eta,$$

with:

$$\begin{aligned} \widehat{f(x - sy)}(\eta) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x - sy) e^{-iy\cdot\eta} dy \\ &= \frac{1}{(2\pi)^n s^n} \int_{\mathbb{R}^n} f(u) e^{-i(x-u)\cdot\eta/s} du = \frac{e^{-ix\cdot\eta/s}}{s^n} \widehat{f}\Big(\frac{-\eta}{s}\Big). \end{aligned}$$

Therefore, since Φ is even:

$$\begin{aligned}\mathbf{f}_s(x) &= \frac{1}{s^n}\int_{\mathbb{R}^n}\Phi(\eta)\widehat{f}\Big(\frac{-\eta}{s}\Big)e^{-ix\cdot\eta/s}d\eta \\ &= \int_{\mathbb{R}^n}\Phi(s\eta)\widehat{f}(-\eta)e^{-ix\cdot\eta}d\eta = \int_{\mathbb{R}^n}\Phi(s\eta)\widehat{f}(\eta)e^{ix\cdot\eta}d\eta.\end{aligned} \tag{17.2.3}$$

Hence $\mathbf{f}_s$ is the inverse Fourier-Laplace transform of the "truncation"

$$\eta \mapsto \Phi(s\eta)\widehat{f}(\eta).$$

The first term of (17.2.2) shows that $\mathbf{f}_s$ extends to $\mathbb{C}^n$ and is an entire function. To get our final estimate we go back to the second term in (17.2.2), which yields

$$|\mathbf{f}_s(z)| \le \|f\|_{C^0(\mathbb{R}^n)}\int_{\mathbb{R}^n}\left|K\left(\frac{z}{s}-y\right)\right|dy, \qquad z\in\mathbb{C}^n. \tag{17.2.4}$$

By the Schwartz estimate of Lemma G.0.1, there exists a constant C_n such that

$$\left|K\left(\frac{z}{s}-y\right)\right| \le C_n\frac{e^{\mathrm{Im}(z/s-y)}}{(1+|z/s-y|_2)^{n+1}},$$

so that, for $y\in\mathbb{R}^n$, $z\in\mathbb{C}^n$ and $|\operatorname{Im} z|_2\le s$:

$$\left|K\left(\frac{z}{s}-y\right)\right| \le C_n\frac{e}{(1+|\operatorname{Re}(z/s-y)|_2)^{n+1}}.$$

Hence:

$$|\mathbf{f}_s(z)| \le \|f\|_{C^0(\mathbb{R}^n)}\,C_n e\int_{\mathbb{R}^n}\frac{dy}{(1+|y|_2)^{n+1}}. \tag{17.2.5}$$

since z/s is fixed and can be eliminated by a simple translation. We finally get the following estimate:

$$|\mathbf{f}_s|_s = \sup_{z\in\mathbb{C}^n:|\operatorname{Im} z|_2\le s}|\mathbf{f}_s(z)| \le C_1(n)\,\|f\|_{C^0(\mathbb{R}^n)}, \tag{17.2.6}$$

with

$$C_1(n) := C_n e\int_{\mathbb{R}^n}\frac{dy}{(1+|y|_2)^{n+1}} < \infty.$$

• *The periodic case.* Fix now an even function $\Psi:\mathbb{R}^n\to[0,1]$, of class C^∞, with support in the ball $\overline{B}_1(0,1)$ and define the associate kernel K as in (17.2.1).

Fix a $2\pi\mathbb{Z}^n$-periodic function $f\in C^\ell(\mathbb{R}^n)$ with $\ell\ge n+1$. Then the Fourier expansion

$$f(\theta) = \sum_{k\in\mathbb{Z}^n}\widehat{f}_k e^{ik\cdot\theta}, \qquad \widehat{f}_k = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}f(\varphi)e^{-ik\cdot\varphi}d\varphi,$$

converges normally since, by Lemma G.0.2 in Appendix G, for $k \in \mathbb{Z}^n \setminus \{0\}$, there exists a universal constant $C_{\mathrm{F}}(n, \ell)$ satisfying

$$\left|\widehat{f}_k\right| \leq C_{\mathrm{F}}(n,\ell)\frac{||f||_{C^{\lfloor \ell \rfloor}}}{|k|_\infty^{\lfloor \ell \rfloor}} \tag{17.2.7}$$

and $\lfloor \ell \rfloor \geq n+1$ by hypothesis. For $s \in]0,1]$, the function

$$\mathbf{f}_s(\theta) = \frac{1}{s^n}\int_{\mathbb{R}^n} K\Big(\frac{\theta-\varphi}{s}\Big) f(\varphi) d\varphi$$

is well-defined and, by the Fubini interversion theorem:

$$\mathbf{f}_s(\theta) = \sum_{k\in\mathbb{Z}^n} \widehat{f}_k \int_{\mathbb{R}^n} K(\varphi) e^{ik\cdot(\theta - s\varphi)} d\varphi = \sum_{k\in\mathbb{Z}^n} \widehat{f}_k e^{ik\cdot\theta} \int_{\mathbb{R}^n} K(\varphi) e^{-isk\cdot\varphi} d\varphi.$$

Hence, since K is the inverse Fourier transform of Ψ, by the Fourier inversion theorem:

$$\mathbf{f}_s(\theta) = \sum_{k\in\mathbb{Z}^n} \widehat{f}_k \Psi(sk)\, e^{ik\cdot\theta}, \quad \theta \in \mathbb{R}^n. \tag{17.2.8}$$

As in the non-periodic case, this makes apparent that $\mathbf{f}_s$ is a continuous truncation of the Fourier expansion of f with a Ψ-dependent modification of its Fourier coefficients (the so-called Jackson polynomial):

$$\widehat{(\mathbf{f}_s)}_k = \Psi(sk)\widehat{f}_k\ . \tag{17.2.9}$$

Consequently, the Fourier norm

$$\left\|\mathbf{f}_s\right\|_s = \sum_{k\in\mathbb{Z}^n} \left|\widehat{(\mathbf{f}_s)}_k\right| e^{s|k|_1}$$

depends only on the harmonics such that $|k|_1 \leq 1/s$ and satisfies

$$\left\|\mathbf{f}_s\right\|_s \leq \sum_{|k|_1\leq 1/s} \left|\widehat{(\mathbf{f}_s)}_k\right| e^{s|k|_1} \leq e \sum_{|k|_1\leq 1/s} \left|\widehat{(\mathbf{f}_s)}_k\right| \leq e \sum_{k\in\mathbb{Z}^n} \left|\widehat{f}_k\right|.$$

Hence, by (17.2.7):

$$\left\|\mathbf{f}_s\right\|_s \leq C_2(\ell)|f|_{C^{\lfloor \ell \rfloor}} \tag{17.2.10}$$

with

$$C_2(\ell) := e\left(1 + C_{\mathrm{F}}(n,\ell) \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|k|_\infty^{\lfloor \ell \rfloor}}\right) \tag{17.2.11}$$

• *Functions on* $\mathbb{A}^n$. We finally gather together the previous two cases. Let $\Phi\otimes\Psi :$ $\mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ be defined by

$$\Phi\otimes\Psi(x,\theta) = \Phi(x)\Psi(\theta),$$

and define the kernel

$$K(y,\varphi)=\int_{\mathbb{R}^{2n}}\Phi\otimes\Psi(x,\theta)\,e^{-i(x,\theta)\cdot(y,\varphi)}\,dxd\theta=K_\Phi(y)K_\Psi(\varphi)=K_\Phi\otimes K_\Psi(y,\varphi)$$

where K_Φ and K_Ψ are defined as above.

Fix a function $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$, $2\pi\mathbb{Z}^n$-periodic with respect to its last n variables, with support in $\overline{B}_2(0,R_0)\times\mathbb{R}^n$ for some $R_0>0$, belonging to $C^\ell(\mathbb{R}^{2n})$ with $\lfloor\ell\rfloor\geq n+1$. For $s\in\,]0,1]$ and $(x,\theta)\in\mathbb{R}^n\times\mathbb{R}^n$, set

$$\begin{aligned}\mathbf{f}_s(x,\theta)&=\int_{\mathbb{R}^{2n}}K(y,\varphi)f(x-sy,\theta-s\varphi)dyd\varphi\\&=\int_{\mathbb{R}^{2n}}K(y,\varphi)\sum_{k\in\mathbb{Z}^n}\widehat{f}_k(x-sy)e^{ik\cdot(\theta-s\varphi)}dyd\varphi\end{aligned}\tag{17.2.12}$$

with

$$\widehat{f}_k(u)=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}f(u,v)e^{-ik\cdot v}dv.\tag{17.2.13}$$

Note that f_k is C^ℓ, with support in $\overline{B}_2(0,R_0)$, so that the previous study on the non-periodic case applies to f_k.

By Fubini interversion

$$\begin{aligned}\mathbf{f}_s(x,\theta)&=\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^{2n}}K(y,\varphi)\widehat{f}_k(x-sy)e^{ik\cdot(\theta-s\varphi)}dyd\varphi\\&=\sum_{k\in\mathbb{Z}^n}\Big(\int_{\mathbb{R}^n}K_\Phi(y)\widehat{f}_k(x-sy)dy\Big)\Big(\int_{\mathbb{R}^n}K_\Psi(\varphi)e^{ik\cdot(\theta-s\varphi)}d\varphi\Big)\\&=\sum_{k\in\mathbb{Z}^n}(\widehat{\mathbf{f}}_k)_s(x)\Psi(sk)e^{ik\cdot\theta}\end{aligned}\tag{17.2.14}$$

where $(\widehat{\mathbf{f}}_k)_s$ stands for the analytic smoothing of the Fourier coefficient $\widehat{f}_k$. This proves that the Fourier coefficient $(\widehat{\mathbf{f}}_s)_k(x)$ relative to the periodic variable θ reads

$$(\widehat{\mathbf{f}}_s)_k(x)=\Psi(sk)(\widehat{\mathbf{f}}_k)_s(x),\quad k\in\mathbb{Z}^n.\tag{17.2.15}$$

Expressions (17.2.14) and (17.2.15) make clear that the whole smoothing procedure of a function depending both on action and angle variables consists in constructing a Jackson trigonometric polynomial by smoothing the Fourier coefficients and by suitably truncating the Fourier series.

Using the definition of Ψ, one has $(\widehat{\mathbf{f}}_s)_k=0$ when $|k|_1>1/s$ and, by (17.2.15) and (17.2.6), one has

$$|(\widehat{\mathbf{f}}_s)_k(z)|\leq|(\widehat{\mathbf{f}}_k)_s(z)|\leq C_1(n)\left\|\widehat{f}_k\right\|_{C^0(\mathbb{R}^n)}\leq C_1(n)C_{\mathrm{F}}(n,\ell)\frac{|f|_{C^{\lfloor\ell\rfloor}(\mathbb{R}^n)}}{|k|_\infty^{\lfloor\ell\rfloor}}\tag{17.2.16}$$

when $k \neq 0,\ |k|_1 \leq 1/s$, whereas

$$|(\widehat{\mathbf{f}}_s)_0(z)| \leq C_1(n) \left\| \widehat{f}_0 \right\|_{C^0(\mathbb{R}^n)} \leq C_1(n) \, \|f\|_{C^0(\mathbb{R}^n)} \, . \tag{17.2.17}$$

As for the weighted Fourier norm of $\mathbf{f}_s$, we finally get:

$$\begin{aligned} ||\mathbf{f}_s||_{s,s} &= \sup_{|\mathrm{Im}\, z|_2 \leq s} \textstyle\sum_{k\in\mathbb{Z}^n} \left|(\widehat{\mathbf{f}}_s)_k(z)\right| e^{s|k|_1} \\ &\leq C_1(n)\, \|f\|_{C^0(\mathbb{R}^n)} + \sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}: \\ |k|_1\leq 1/s}} eC_1(n)C_{\mathrm{F}}(n,\ell)\frac{|f|_{C^{\lfloor \ell\rfloor}(\mathbb{R}^n)}}{|k|_\infty^{\lfloor \ell\rfloor}} \\ &\leq C_L(n,\ell)|f|_{C^\ell(\mathbb{R}^n)}\, , \end{aligned}$$

where

$$C_L(n,\ell) := C_1(n)\left(1 + eC_{\mathrm{F}}(n,\ell)\sum_{k\in\mathbb{Z}^n}\frac{1}{|k|_\infty^{\lfloor \ell\rfloor}}\right) < +\infty. \tag{17.2.18}$$

17.3 The main result with an application to normal forms.

17.3.1 Main result

Gathering together the elements of the previous section, we get the following result.

Theorem 17.3.1 (Analytic smoothing)**.** *Fix an integer $n \geq 1$, $R > 0$ and $s \in]0,1]$. Let f be a C^ℓ function on $B_\infty(0,2R)\times\mathbb{T}^n$. There exist two constants $C_A(R,\ell,n)$, $C_B(R,\ell,n)$ and an analytic function $\mathtt{f}_s$ on the set $\mathbb{A}^n_s$ satisfying*

$$\|f - \mathcal{L}_s(f)\|_{C^p(\mathtt{B}^R\times\mathbb{T}^n)} \leq \ C_A(R,\ell,n)\ s^{\ell-p}|f|_{C^\ell(B_\infty(0,2R)\times\mathbb{T}^n)} \tag{17.3.1}$$

for any integer $0 \leq p \leq \lfloor \ell \rfloor$, and

$$\|\mathtt{f}_s\|_{s,s} \leq C_B(R,\ell,n)|f|_{C^\ell(B_\infty(0,2R)\times\mathbb{T}^n)}. \tag{17.3.2}$$

Moreover, $\mathcal{L}_s(f)$ is a trigonometric polynomial in the angular variables.

Proof. Fix a function $\chi \in C^\infty(\mathbb{R}^n)$, with values in $[0,1]$, equal to 1 on the ball $\mathtt{B}^R$ and with support in $\mathtt{B}^{2R}$. Then the product $\overline{f} := \chi f$ is C^ℓ on $\mathbb{A}^n$, has compact support in $\mathtt{B}^{2R}\times\mathbb{T}^n$ and coincides with f on $\mathtt{B}^R\times\mathbb{T}^n$. Moreover

$$|\overline{f}|_{C^\ell(B_\infty(0,2R)\times\mathbb{T}^n)} \leq \mathtt{C}_K |f|_{C^\ell(B_\infty(0,2R)\times\mathbb{T}^n)}$$

where $\mathtt{C}_K = C|\chi|_{C^\ell(\mathtt{B}^R\times\mathbb{T}^n)}$ and C is a universal constant. By the Jackson-Moser-Zehnder theorem applied to $\overline{f}$, there is an analytic function $\mathcal{L}_s(\bar f)$ on $\mathbb{A}^n_s$ satisfying

$$\left|\partial^\alpha\mathcal{L}_s(\bar f)(I,\theta) - \sum_{\substack{\beta\in\mathbb{N}^{2n}: \\ |\beta|\leq\lfloor \ell\rfloor-|\alpha|}} \partial^{\alpha+\beta}\bar f(\mathrm{Re}(I,\theta))\frac{(\mathrm{Im}(I,\theta))^\beta}{\beta!}\right| \leq C_{\mathtt{J}} s^{\ell-|\alpha|}|\bar f|_{C^\ell(\mathbb{A}^n)}, \tag{17.3.3}$$

so that for any $p \leq \lfloor \ell \rfloor$:

$$\left\|\bar{f} - \mathcal{L}_s(\bar{f})\right\|_{C^p(\mathbb{A}^n)} \leq C_{\mathrm{J}} s^{\ell - p} |\bar{f}|_{C^\ell(\mathbb{A}^n)}. \tag{17.3.4}$$

As a consequence, taking the form of χ into account, one gets

$$\left\|f - \mathcal{L}_s(f)\right\|_{C^p(\mathbb{B}^R \times \mathbb{T}^n)} \leq \mathtt{C}_K C_{\mathrm{J}} s^{\ell - p} |f|_{C^\ell(B_\infty(0,2R) \times \mathbb{T}^n)}. \tag{17.3.5}$$

Setting $C_A := \mathtt{C}_K C_{\mathrm{J}}$ and, since the analyticity width ρ of the integrable part h is greater than s, the bound (17.3.1) follows. The proof of (17.3.2) is an immediate consequence of the previous paragraphs if one sets $C_B := C_L \times C_K$. □

17.3.2 An easy way to derive normal forms for Hölder functions from analytic ones.

Let us now explain our strategy for a general Hölder Hamiltonian, we will then restrict ourselves to the case where h is analytic. Let

$$H(I, \theta) := h(I) + f(I, \theta) \tag{17.3.6}$$

be C^ℓ on $B_\infty(0, 2R) \times \mathbb{T}^n$. Given $s \in]0, 1]$, let $\mathbf{H}_s$ be the s-smoothed analytic function given by Theorem 17.3.1 applied to the function H. By classical constructions (alluded to in the introduction and which will be recalled in the following), there exist (close to identity) symplectic analytic local diffeomorphisms Ψ defined on domains $D \subset \mathbb{A}^n$ which bring $\mathbf{H}_s = \mathbf{h}_s + \mathbf{f}_s$ to the normal form $\mathbf{H}_s \circ \Psi : D \to \mathbb{R}$:

$$\mathbf{H}_s \circ \Psi = \mathbf{h}_s + \mathbf{g} + \mathbf{f}_{\mathbf{s}}^* \tag{17.3.7}$$

where $\mathbf{h}_s$ is nothing else than the smoothed initial integrable Hamiltonian, g is a resonant part which controls the fast drift in certain directions and $\mathbf{f}_s^*$ is a very small remainder – all these functions being analytic on D. The keypoint in our subsequent constructions is the following very simple equality

$$H \circ \Psi = \mathbf{H}_s \circ \Psi + (H - \mathbf{H}_s) \circ \Psi = \mathbf{h}_s + \mathbf{g} + \left[\mathbf{f}_{\mathbf{s}}^* + (\mathbf{H} - \mathbf{H}_{\mathbf{s}}) \circ \boldsymbol{\Psi}\right]. \tag{17.3.8}$$

This is a normal form for H, obtained by composition of H with an *analytic* diffeomorphism, in which the first three terms are analytic on D and only the last one is C^ℓ. So $H \circ \Psi$ has the same structure and dynamical interpretation as $\mathbf{H}_s \circ \Psi$, *provided that the* C^ℓ *size of the additional remainder* $(H - \mathbf{H}_s) \circ \Psi$ *is of the same order as the size of the initial remainder* $\mathbf{f}_s^*$. This issue strongly depends on the analytic smoothing method in use, we will show in the sequel that the Jackson-Moser-Zehnder method is relevant for our purposes. Our study will be even easier since we assume from the beginning that the integrable part h is analytic.

It turns out that the same smoothing method - and the same simple way to get a normal form from an analytical one - are also relevant in many other functional classes, the main ones being the Gevrey classes already used in [87], but also other ultradifferentiable ones. This will be developed in a further work.

Chapter 18

Estimates of stability

The aim of this section is to prove Theorem 14.3.1. The proof consists of several steps. Following the discussion in section 15.1 of the introduction, we first build an appropriate resonant covering of the phase space for the integrable Hamiltonian h. Secondly, we study the local dynamics by applying Pöschel's resonant normal form (see Appendix H) in each resonant block and we set the dependencies of the ultraviolet cut-off K and analyticity widths r, s on the perturbative parameter ε. Finally, we exploit the properties of the resonant covering and we obtain a global result of stability by exploiting the so called "capture in resonance" argument.

18.1 Construction of the resonant patchwork

In the sequel, we follow ref. [70], in which the choices of the parameters and the dependencies of the small denominators on the ultraviolet cut-off K are justified heuristically. For the sake of clarity, in order to have coherent notations we denote by D_Λ rather than $\mathcal{B}_\Lambda$ the resonant blocks introduced in Section 15, moreover when possible we will not keep track of constants [1] but rather indicate their presence in bounds and equalities by using the following symbols respectively: $\mathrel{\mathring{=}}$, $\lessdot$ and $\gtrdot$.

We start by setting some parameters, depending on the steepness indices $\boldsymbol{\alpha}_1, .., \boldsymbol{\alpha}_{n-1}$ of h, that will be useful throughout this section.

$$p_j := \begin{cases} \Pi_{i=j}^{n-2} \boldsymbol{\alpha}_i & , \quad \text{if } j \in \{1, ..., n-2\} \\ 1 & , \quad \text{if } j \in \{n-1, n\} \end{cases} \tag{18.1.1}$$

$$q_j := np_j - j \;, \quad j \in \{1, ..., n\} \;; \qquad c_j := q_j - q_{j+1} \;, \quad j \in \{1, ..., n-1\}$$

[1] i.e. of quantities depending only on the fixed parameters of the problem, namely n, h, ℓ and on the indices of steepness $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_{n-1}$.

and set

$$a := \frac{1}{2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-2}} = \frac{1}{2np_1} \quad , \qquad b := \frac{1}{2n\boldsymbol{\alpha}_1...\boldsymbol{\alpha}_{n-1}} = \frac{a}{\boldsymbol{\alpha}_{n-1}} \quad , \qquad \mathrm{R}(\varepsilon) :\mathring{=} \varepsilon^b \ . \tag{18.1.2}$$

With this setting, we fix an action $I_0 \in B_\infty(0, R/4)$ and we consider its neighborhood $B_2(I_0, \mathrm{R}(\varepsilon))$.

Since h is steep in $B_\infty(0, R)$, the norm of the frequency $\omega := \partial_I h(I)$ at any point of this set admits a uniform lower positive bound, that is $\inf_{I\in B_\infty(0,R)} ||\omega(I)|| \gg 1$. Hence, when studying the geography of resonances for h, for sufficiently small ε and without any loss of generality we can just consider maximal lattices $\Lambda \subset \mathbb{Z}^n_K$ of dimension $j \in \{0, ..., n-1\}$, with $K \geq 1$ the ultraviolet cut-off. For a lattice Λ of dimension $j \in \{0, ..., n-1\}$ we define its associated *resonant zone* as

$$Z_\Lambda := \{I \in B_2(I_0, \mathrm{R}(\varepsilon)) : \ \forall k \in \Lambda \text{ one has } \ |k \cdot \omega(I)| < \delta_\Lambda\} \ , \ \ \delta_\Lambda :\mathring{=} \frac{1}{|\Lambda| K^{q_j}} \ . \tag{18.1.3}$$

and its associated *resonant block* D_Λ as

$$D_\Lambda := Z_\Lambda \backslash \bigcup_{\Lambda' : \dim \Lambda' = j+1} Z_{\Lambda'} \ . \tag{18.1.4}$$

Note that D_Λ corresponds to that part of the resonant zone Z_Λ which does not contain any other resonances other than the one associated to Λ. In particular, this implies that for the completely non-resonant block associated to $\Lambda = \{0\}$ and for any block Λ corresponding to a maximal resonance of dimension $j = n-1$ one has, respectively

$$D_0 := B(I_0, \mathrm{R}(\varepsilon)) \backslash \bigcup_{\Lambda' : \ \dim \Lambda' = 1} Z_{\Lambda'} \quad \text{and} \quad D_\Lambda = Z_\Lambda \ . \tag{18.1.5}$$

For any $j \in \{0, ..., n-1\}$ we set

$$D_j := \bigcup_{\Lambda : \ \dim \Lambda = j} D_\Lambda \quad , \qquad Z_j := \bigcup_{\Lambda : \ \dim \Lambda = j} Z_\Lambda \ . \tag{18.1.6}$$

It is easy to see from (18.1.4) that

$$D_j = Z_j \backslash Z_{j+1} \tag{18.1.7}$$

so that from the definition of D_0 in (18.1.5) one has the decompositions

$$B_2(I_0, \mathrm{R}(\varepsilon)) = \bigcup_{i=0}^{n-1} D_i \quad , \qquad B_2(I_0, \mathrm{R}(\varepsilon)) = \left(\bigcup_{i=0}^{j-1} D_i \right) \cup Z_j \qquad \forall j = 1, ..., n-1 \ . \tag{18.1.8}$$

As we have explained in the introduction (see section 15.1), a large drift over a short time of any action variable $I \in D_\Lambda$ is only possible along the plane of fast drift $I + \langle \Lambda \rangle$ spanned by the vectors belonging to Λ. Moreover, the fast motion of the orbit starting at I along $I + \langle\Lambda\rangle$ can take the actions out of the block D_Λ. So, we are interested in understanding what happens when the actions leave D_Λ but keep staying in Z_Λ. Hence, we are naturally taken to consider the intersection of a neighborhood of $I + \langle\Lambda\rangle$ with Z_Λ. In this spirit, we fix

$$\rho(\varepsilon) := \frac{\mathtt{R}(\varepsilon)}{2n} \tag{18.1.9}$$

and, for any $0 < \eta \leq \rho(\varepsilon)$ and for any action $I \in D_\Lambda$ with $\Lambda \neq \{0\}$, we define the *disc* associated to I as

$$\mathbf{D}^{\rho}_{\Lambda,\eta}(I) := \left(\left(\bigcup_{I' \in I + \langle\Lambda\rangle} B_2(I', \eta) \right) \cap Z_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)\big) \right)_I \tag{18.1.10}$$

where the subscript I denotes the connected component of the set containing the action I. Since we are going to study the fate of all orbits starting at a fixed block D_Λ, with $\Lambda \neq \{0\}$, that exit such block in a short time along the plane of fast drift, we are also led to define the *extended resonant block*

$$D^{\rho}_{\Lambda,r_\Lambda} := \left(\bigcup_{I \in D_\Lambda \cap B(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon))} \mathbf{D}^{\rho}_{\Lambda,r_\Lambda}(I) \right) \subset Z_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)\big) , \quad r_\Lambda := \frac{\delta_\Lambda}{M} , \tag{18.1.11}$$

where M was defined in (14.3.3). In the same way, the *extended non-resonant block* is defined as

$$D^{\rho}_{0} := D_0 \cap B(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)) . \tag{18.1.12}$$

18.1.1 The resonant blocks

As we have explained there, Nekhoroshev proved in [94] that, if h is steep, when any action $I \in D_\Lambda$, with $\Lambda \neq \{0\}$, moves along the plane of fast drift, it must exit the resonant zone Z_Λ after having travelled for a short distance. Indeed, if h is steep with steepness indices $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_{n-1}$ one can prove that the diameter of the intersection of a neighborhood of the fast drift plane with the resonant zone is small in the sense given by the following

Lemma 18.1.1. *For any* $\Lambda \neq 0$, $\dim\Lambda = j \in \{1, ..., n-1\}$, *for any* $I \in D_\Lambda \cap B(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon))$ *and for any* $I' \in \mathbf{D}^{\rho}_{\Lambda,r_\Lambda}(I)$ *one has*

$$\left|I - I'\right|_2 \leq r_j \quad , \qquad \textit{where} \quad r_j :\stackrel{\circ}{=} \frac{1}{K^{q_j/\alpha_j}} . \tag{18.1.13}$$

For a proof of this result we refer to Lemma 2.1 of ref. [70].

We notice that a smaller value of ε, i.e. a higher value of K since the ultraviolet cut-off is always a decreasing function of ε, leads to a closer maximal distance between any action I belonging to a resonant block and any action belonging to its disc.

Since we will perform normal forms in the (extended) resonant blocks, we also need an estimate of the small divisors in these sets, namely we have

Lemma 18.1.2. *For any maximal lattice $\Lambda \in \mathbb{Z}^n_K$ of dimension $j \in \{0, ..., n-1\}$, for any $k \in \mathbb{Z}^n_K \backslash \Lambda$ and for any $I \in D^\rho_{\Lambda, r_\Lambda}$ one has*

$$|\langle k, \omega(I)\rangle| \geq \alpha_\Lambda :\circeq \frac{1}{|\Lambda| K^{q_j - c_j}} \ , \tag{18.1.14}$$

whereas for any action I in the completely non-resonant block D_0 and for any $k \in \mathbb{Z}^n_K$ one has

$$|\langle k, \omega(I)\rangle| \geq \alpha_0 :\circeq \frac{1}{K^{q_1}} \ . \tag{18.1.15}$$

We refer again to [70, Lemma 2.2] for a proof of this result.

Finally, a key ingredient in order to insure stability in the steep case is the fact that, when possibly exiting a resonant zone along the plane of fast drift, the actions must enter another resonant zone associated to a lattice of lower dimension. This is the content of

Lemma 18.1.3. *Let Λ, Λ' two maximal lattices of $\mathbb{Z}^n_K$ having the same dimension $j \in \{1, ..., n-1\}$. Then one has*

$$\textit{closure}\left(D^\rho_{\Lambda, r_\Lambda}\right) \cap Z_{\Lambda'} = \varnothing \ . \tag{18.1.16}$$

Once again, the proof of this Lemma can be found in [70] (Lemma 2.3).

With the ingredients of this paragraph, we are able to prove stability.

18.2 Proof of the Main Theorem

In order to prove Theorem 14.3.1, we start by giving the standard estimates of stability in the completely non-resonant extended block D^ρ_0. Note that the following bounds do not require any geometric assumption on the integrable part h.

Lemma 18.2.1 (Non-resonant Stability Estimates)**.** *For any sufficiently small ε and for any time t satisfying*

$$|t| \leq T_0 :\circeq \frac{1}{(1 + a\ell)|\ln \varepsilon|^{\ell - 1}\ \varepsilon^{a(\ell-1)+1/2}} \quad , \qquad a := \frac{1}{2np_1} \ , \tag{18.2.1}$$

any initial condition $I(0) \in D^\rho_0$ drifts at most as

$$|I(t) - I(0)|_2 \lesssim \varepsilon^{1/2} \ . \tag{18.2.2}$$

Proof. Our goal is to apply Pöschel's normal form (see Lemma H.0.1) to the smoothed Hamiltonian of Theorem 17.3.1 with analyticity widths r and s.

• **Normal form**

By monotonicity of the Fourier norm w.r.t. the action variables and (17.3.2) we immediately get,

$$||\mathcal{L}_s(f)||_{r,s} \leq ||\mathcal{L}_s(f)||_{s,s} \leq C_B(R, \ell, n)\varepsilon =: \epsilon \ , \tag{18.2.3}$$

for any $r \leq s$, where we set $\varepsilon := |f|_{C^{\ell}(\mathbb{B}^R\times\mathbb{T}^n)}$.

Denote

$$\mathcal{B}_{\varrho,\sigma} := \{(I,\theta)\in\mathbb{C}^n \,:\, |I - B_{\infty}(0,R/4)|_2 < \varrho\,,\;\; \theta\in\mathbb{T}^n_{\sigma}\}\;,$$

since h is analytic, we chose not to regularize it further. So let $\mathtt{H}_s := h(I) + \mathtt{f}_s$ be the corresponding analytic Hamiltonian defined on $\mathcal{B}_{s,s}$. By Pöschel's Lemma H.0.1 applied in the complex extension, denoted $\mathcal{D}^{\rho}_{0,r,s}$, of the non-resonant block D^{ρ}_{0}, with $\varrho' \rightsquigarrow r, \varrho \rightsquigarrow s, \sigma \rightsquigarrow s$, if

$$\epsilon \lessapprox \frac{\alpha_0\, r}{K}\;,\quad r \lessapprox \min\left(\frac{\alpha_0}{K}, s\right)\;,\quad Ks \geq 6 \tag{18.2.4}$$

are satisfied, then there exists a symplectic diffeomorfism Ψ_0 that puts $\mathtt{H}_s$ into resonant normal form:

$$\mathtt{H}_s\circ\Psi_0 = h(I) + \mathtt{g} + \mathtt{f}^*_s\;,\;\; \{h,\mathtt{g}\} = 0, \quad \Psi_0 \,:\, \mathcal{D}^{\rho}_{0,r/2,s/6} \longrightarrow \mathcal{D}^{\rho}_{0,r,s}\;. \tag{18.2.5}$$

In particular the resonant and non-resonant part satisfy, respectively,

$$||\mathtt{g} - \mathbf{g_0}||_{r/2,s/6} \lessapprox \epsilon\;,\;\; ||\mathtt{f}^*_s||_{r/2,s/6} \leq e^{-\frac{Ks}{6}}\epsilon \tag{18.2.6}$$

where $\mathbf{g_0} := P_{\Lambda}P_K\mathcal{L}_s(f)$ and P_{Λ}, P_K are the projectors defined in Lemma H.0.1.

• **Setting of the initial parameters**

Let us set the following dependences on ϵ of the ultraviolet cut-off K and of the analyticity widths r, s

$$\begin{aligned} K &:= \left(\frac{\epsilon_0}{\epsilon}\right)^a \quad , \qquad s :\mathrel{\hat{=}} \left(\frac{\epsilon}{\epsilon_0}\right)^a \left|\ln\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{6(1+a\ell)}\right]\right| \\ r &:\mathrel{\hat{=}} \frac{1}{K^{1+q_1}} \mathrel{\hat{=}} \left(\frac{\epsilon}{\epsilon_0}\right)^{a(1+q_1)} = \left(\frac{\epsilon}{\epsilon_0}\right)^{1/2}\;. \end{aligned} \tag{18.2.7}$$

where ϵ_0 is a free parameter and $\epsilon \leq \epsilon_0$ since $K \geq 1$.

Remark 18.2.1. The freedom in the definitions above is subordinated to the fact that, in order for the construction to be meaningful, the reminder produced by the normal form must be less than or equal to the size of the additional term $(H - \mathcal{L}_s(H))\circ\Psi_0$, byproduct of the analytic smoothing. As we are working in finite regularity, the latter is expected to be polynomial. The reminder of the normal form being of order e^{-Ks}, one must have $Ks \sim O(|\log\epsilon|^c)$ for some $c > 0$. Since s tunes the size of the remainder yielded by the analytic smoothing, it has to be polynomial. Hence one is left with two possibilities: either the choice we made in (18.2.7), or to set $K \sim \epsilon^{-a}|\log\epsilon|^c$ and $s \sim \epsilon^a$. However this second choice would worsen the exponents of stability, since the thresholds of applicability in the normal form lemma strongly depend on K. Of course, to deal with other regularity classes, such as the Gevrey one, other choices must be made.

By plugging the choices (18.2.7) into the three thresholds in (18.2.4), it is easy to see that there exists an appropriate choice of ϵ_0 that makes the three conditions to be simultaneously satisfied. Hence, for the Hölder Hamiltonian

$$H = h + f = \mathtt{H}_s + f - \mathtt{f}_s, \qquad \mathtt{H}_s := h + \mathtt{f}_s$$

we can write

$$H \circ \Psi_0 = \mathcal{L}_s(H) \circ \Psi_0 + (f - \mathcal{L}_s(f)) \circ \Psi_0 = h + \mathtt{f}_s^* + (f - \mathcal{L}_s(f)) \circ \Psi_0 . \qquad (18.2.8)$$

Note that since we are in a completely non-resonant block, the resonant term g does not appear in the normal form. Now, the normal form in Lemma H.0.1 insures that there exists a constant $\xi > 1$ such that any initial condition $(I(0), \theta(0)) \in D_0^\rho \times \mathbb{T}^n$ is mapped by Ψ_0 into $(\mathtt{I}(0), \vartheta(0)) \in (\mathcal{D}^\rho_{0,\frac{r}{32\xi}})^{\mathbb{R}} \times \mathbb{T}^n$. For any time t such that the normalized flow $\Phi^t_{H\circ\Psi_0} : (\mathtt{I}(0), \vartheta(0)) \longmapsto (\mathtt{I}(t), \vartheta(t))$ starting at $(\mathcal{D}^\rho_{0,\frac{r}{32\xi}})^{\mathbb{R}} \times \mathbb{T}^n$ does not exit from $(\mathcal{D}^\rho_{0,r/2})^{\mathbb{R}} \times \mathbb{T}^n$, the evolution of the normalized variables reads ($i = 1, ..., n$)

$$\begin{aligned}
&|\mathtt{I}_i(t) - \mathtt{I}_i(0)| \\
&\le \int_0^t \sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^\rho_{0,\frac{r}{32\xi}})^{\mathbb{R}}\times\mathbb{T}^n} \left(\left|(\partial_{\vartheta_i} \mathtt{f}_s^*)\circ\Phi^t_{H\circ\Psi_0}\right| + \left|\{\partial_{\vartheta_i}[(f - \mathcal{L}_s(f))\circ\Psi_0]\}\circ\Phi^t_{H\circ\Psi_0}\right| \right) dt \\
&\le \int_0^t \left(\sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^\rho_{0,r/2})^{\mathbb{R}}\times\mathbb{T}^n} |\partial_{\vartheta_i} \mathtt{f}_s^*| + \sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^\rho_{0,r/2})^{\mathbb{R}}\times\mathbb{T}^n} |\partial_{\vartheta_i}[(f - \mathcal{L}_s(f))\circ\Psi_0]| \right) dt \\
&\le |t| \left[\frac{||\mathcal{L}_s(f)^*||_{r/2,s/6}}{s} + \left\| f - \mathcal{L}_s(f) \right\|_{C^1(B_\infty(0,R/2)\times\mathbb{T}^n)} \times \sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^\rho_{0,r/2})^{\mathbb{R}}\times\mathbb{T}^n} |\partial_{\vartheta_i}\Psi_0| \right] .
\end{aligned} \qquad (18.2.9)$$

The normal form Lemma H.0.1, together with the choices in (18.2.7) and the definition of ϵ in (18.2.3), assures that

$$||\mathcal{L}_s(f)^*||_{r/2,s/6} \le e^{-Ks/6}\,\epsilon \lesssim \exp\left\{ \ln\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{1+a\ell} \right] \right\} \epsilon \lesssim \varepsilon^{2+a\ell} , \qquad (18.2.10)$$

whereas, by Theorem 17.3.1, we have

$$\begin{aligned}
&\left\| f - \mathcal{L}_s(f) \right\|_{C^1(B_\infty(0,R/2)\times\mathbb{T}^n)} \\
&\lesssim s^{\ell-1}\varepsilon \lesssim \left| \ln\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{6(1+a\ell)} \right] \right|^{\ell-1} \varepsilon^{1+a(\ell-1)} \lesssim \left| \ln\left(\varepsilon^{6(1+a\ell)} \right) \right|^{\ell-1} \varepsilon^{1+a(\ell-1)} .
\end{aligned} \qquad (18.2.11)$$

Finally, by writing in the usual way $|\partial_{\vartheta_i}\Psi_0| = |\partial_{\vartheta_i}(\Psi_0 - \mathrm{id} + \mathrm{id})|$, the Cauchy estimates together with the bounds in H.0.5 imply (since $r \le s$)

$$\sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^{\rho}_{0,r/2})^{\mathbb{R}}\times\mathbb{T}^n} |\partial_{\vartheta_i}\Psi_0|_2 \lesssim 1 + \max\left\{\frac{1}{24\xi}, \frac{1}{32\xi}\frac{r}{s}\right\} \lesssim 1 \; . \tag{18.2.12}$$

It is easy to see from estimates (18.2.10), (18.2.11) and (18.2.12) that, in order, the remainder from the analytic smoothing dominates on the one coming from the normal form, namely

$$\frac{||\mathcal{L}_s(f)^*||_{r/2,s/6}}{s} \ll \left\|f - \mathcal{L}_s(f)\right\|_{C^1(B_\infty(0,R/2)\times\mathbb{T}^n)} \times \sup_{(\mathtt{I},\vartheta)\in(\mathcal{D}^{\rho}_{0,r/2})^{\mathbb{R}}\times\mathbb{T}^n} |\partial_{\vartheta_i}\Psi_0|$$

so that finally we can write

$$|\mathtt{I}(t) - \mathtt{I}(0)|_2 \lesssim |t| \left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1+a(\ell-1)} \; . \tag{18.2.13}$$

Hence, over a time

$$|t| \lesssim \frac{r}{\left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1+a(\ell-1)}} \lesssim \frac{1}{\left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1/2+a(\ell-1)}}$$

one has $|\mathtt{I}(t) - \mathtt{I}(0)|_2 \lesssim r$ and, by scaling back to the original variables,

$$|I(t) - I(0)|_2 \lesssim r \lesssim \varepsilon^{1/2} \; .$$

□

As for the dynamics in the resonant blocks, we have the following

Lemma 18.2.2. *Consider a maximal lattice $\Lambda \subset \mathbb{Z}^n_K$ of dimension $j \in \{1, ..., n-1\}$. There exists $\mathtt{T}_j > 0$ such that for any sufficiently small ε and for any initial condition $(I(0), \theta(0)) \in \left(D_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - (j+1)\rho(\varepsilon)\big)\right) \times \mathbb{T}^n$, if one sets*

$$T_\Lambda := \mathtt{T}_j \times \frac{r_\Lambda}{|\ln \varepsilon^{6(1+a\ell)}|^{\ell-1}\, \varepsilon^{1+a(\ell-1)}} \quad , \qquad a := \frac{1}{2np_1} \; , \tag{18.2.14}$$

and considers the time of escape of the flow generated by H from the extended resonant block

$$\tau_e := \inf\left\{ t \in \mathbb{R} \, : \, \Phi^t_H\left(D_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - (j+1)\rho(\varepsilon)\big) \times \mathbb{T}^n\right) \not\subset D^{\rho}_{\Lambda, r_\Lambda} \times \mathbb{T}^n \right\} \; , \tag{18.2.15}$$

the following dichotomy applies:

1. *If* $|\tau_e| \geq T_\Lambda$ *one has*

$$|I(t) - I(0)|_2 < \rho(\varepsilon) \tag{18.2.16}$$

over a time $|t| \leq T_\Lambda$*;*

2. *If* $|\tau_e| < T_\Lambda$ *there exists* $i \in \{0, ..., j-1\}$ *such that*

$$I(\tau_e) \in D_i \cap \left(B\big(I_0, \mathtt{R}(\varepsilon) - j\rho(\varepsilon)\big) \right) .$$

Proof. We start by considering the case $|\tau_e| \geq T_\Lambda$. In a similar way to what we did in the proof of Lemma 18.2.1, we apply Pöschel's Normal Form (see Lemma H.0.1) to the smoothed Hamiltonian $\mathtt{H}_s$ in the complex extension $(D^\rho_{\Lambda, r_\Lambda})_{r_\Lambda}$ of the real extended resonant block $D^\rho_{\Lambda, r_\Lambda}$, with parameters

$$K := \left(\frac{\epsilon_0}{\epsilon}\right)^a \ , \quad s :\doteq \left(\frac{\epsilon}{\epsilon_0}\right)^a \left| \ln\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{6(1+a\ell)} \right] \right| \ , \quad r_\Lambda :\doteq \frac{1}{|\Lambda| K^{q_j}} \tag{18.2.17}$$

and with a small divisor estimate given by formula (18.1.14) in Lemma 18.1.2, namely

$$\alpha_\Lambda :\doteq \frac{1}{|\Lambda| K^{q_j - c_j}} \ . \tag{18.2.18}$$

As before, we plug (18.2.17) and (18.2.18) into Pöschel's thresholds (H.0.1) – (H.0.2) and we derive the conditions

$$\begin{aligned} \epsilon \lesssim \frac{\alpha_\Lambda r_\Lambda}{K} \quad &\longleftrightarrow \quad \left(\frac{\epsilon}{\epsilon_0}\right)^{1-an(p_j+p_{j+1})} \lesssim 1 \qquad j \in \{1, \dots, n-1\} \\ r_\Lambda \lesssim \frac{\alpha_\Lambda}{K} \quad &\longleftrightarrow \quad \left(\frac{\epsilon}{\epsilon_0}\right)^{an(p_j - p_{j+1})} \lesssim 1 \qquad j \in \{1, \dots, n-1\} \\ Ks \geq 6 \quad &\longleftrightarrow \quad \left| \ln\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{6(1+a\ell)} \right] \right| \geq 6 \ . \end{aligned} \tag{18.2.19}$$

By definition of the parameters p_j in (18.1.1), it is easy to see that the first two conditions are always satisfied by appropriately choosing ϵ_0, whereas the last condition is trivial.

Therefore, by taking into account the notations in (16.2.3), there exists a symplectic transformation $\Psi_\Lambda : (D^\rho_{\Lambda, r_\Lambda})_{r_\Lambda/2} \times \mathbb{T}^n_{s/6} \longrightarrow (D^\rho_{\Lambda, r_\Lambda})_{r_\Lambda} \times \mathbb{T}^n_s$, $(\mathtt{I}, \vartheta) \longmapsto (I, \theta)$, that takes H into the resonant normal form

$$H \circ \Psi_\Lambda = \mathcal{L}_s(H) \circ \Psi_\Lambda + (H - \mathcal{L}_s(H)) \circ \Psi_\Lambda = h + \mathtt{g} + \mathtt{f}^*_s + (f - \mathcal{L}_s(f)) \circ \Psi_\Lambda \tag{18.2.20}$$

with $\{h, \mathtt{g}\} = 0$, $||\mathtt{f}^*_s||_{r/2, s/6} \lesssim e^{-Ks/6}\, \varepsilon$.

Now, for any time t such that $|t| \leq T_\Lambda \leq |\tau_e|$, the dynamics on the subspace orthogonal to the plane of fast drift $\langle \Lambda \rangle$ can be controlled in the usual way by exploiting the smallness of the non-resonant remainder $\mathtt{f}^*_s$, as well as that of $(f - \mathtt{f}_s) \circ \Psi_\Lambda$. Namely, for any initial position in the actions $I(0) \in D_\Lambda$, by the first estimate in (H.0.5) one has that the associated normalized coordinate satisfies $\mathtt{I}(0) \in (D_\Lambda)^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}}$, where $(D_\Lambda)^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}}$ represents the real projection of the complex extension of width $\frac{r_\Lambda}{32\xi}$ around D_Λ (not to be confused with the extended resonant block) and where $\xi > 1$ is a free parameter that can be suitably adjusted. By taking into account the fact that $\Pi_{\langle\Lambda\rangle^\perp}(\partial_\vartheta \mathtt{g}) = 0$, one can write

$$\begin{aligned}
&\left|\Pi_{\langle\Lambda\rangle^\perp}\big(\mathtt{I}(t) - \mathtt{I}(0)\big)\right|_2 \\
&\leq \int_0^t \sup_{(\mathtt{I},\vartheta)\in (D_\Lambda)^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}} \times \mathbb{T}^n} \Bigg(\left|\Pi_{\langle\Lambda\rangle^\perp}(\partial_\vartheta \mathtt{g} + \partial_\vartheta \mathtt{f}^*_s)\circ\Phi^t_{H\circ\Psi_\Lambda}\right|_2 \\
&\qquad\qquad + \left|\Pi_{\langle\Lambda\rangle^\perp}\{\partial_\vartheta[(f - \mathcal{L}_s(f))\circ\Psi_\Lambda]\}\circ\Phi^t_{H\circ\Psi_\Lambda}\right|_2 \Bigg) dt \\
&\leq \int_0^t \sup_{(\mathtt{I},\vartheta)\in (D_\Lambda)^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}} \times \mathbb{T}^n} \Bigg(\left|(\partial_\vartheta \mathtt{f}^*_s)\circ\Phi^t_{H\circ\Psi_\Lambda}\right|_2 + \left|\{\partial_\vartheta[(f - \mathcal{L}_s(f))\circ\Psi_\Lambda]\}\circ\Phi^t_{H\circ\Psi_\Lambda}\right|_2 \Bigg) dt \\
&\leq \sup_{(\mathtt{I},\vartheta)\in \left(D^\rho_{\Lambda,r_\Lambda}\right)^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}} \times \mathbb{T}^n} \Bigg(\left|(\partial_\vartheta \mathtt{f}^*_s)\right|_2 + \left|\{\partial_\vartheta[(f - \mathcal{L}_s(f))\circ\Psi_\Lambda]\}\right|_2 \Bigg) |t| \ ,
\end{aligned} \tag{18.2.21}$$

where the last inequality follows from the fact that $|t| \leq \tau_e$ and, since the initial variables are confined in $D^\rho_{\Lambda,r_\Lambda}$, the normalized ones stay in $(D^\rho_{\Lambda,r_\Lambda})^{\mathbb{R}}_{\frac{r_\Lambda}{32\xi}}$ over the same time.

Since $|t| \leq T_\Lambda \leq \tau_e$, by the same arguments that were used in estimate (18.2.9) and estimate (18.2.21) we obtain

$$\begin{aligned}
&\left|\Pi_{\langle\Lambda\rangle^\perp}\big(\mathtt{I}(t) - \mathtt{I}(0)\big)\right|_2 \\
&\lesssim |t| \left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1+a(\ell-1)} \\
&\lesssim \mathtt{T}_j \times \frac{r_\Lambda}{\left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1+a(\ell-1)}} \left|\ln\left(\varepsilon^{6(1+a\ell)}\right)\right|^{\ell-1} \varepsilon^{1+a(\ell-1)} = \frac{r_\Lambda}{4}
\end{aligned} \tag{18.2.22}$$

by suitably choosing $\mathtt{T}_j$.

Let us decompose the variation of the action variables as

$$\begin{aligned}
I(t) - I(0) =& I(t) - \mathtt{I}(t) + \mathtt{I}(t) - \mathtt{I}(0) + \mathtt{I}(0) - I(0) \\
=& I(t) - \mathtt{I}(t) + \Pi_{\langle\Lambda\rangle^\perp}\big(\mathtt{I}(t) - \mathtt{I}(0)\big) + \Pi_{\langle\Lambda\rangle}\big(\mathtt{I}(t) - \mathtt{I}(0)\big) + \mathtt{I}(0) - I(0) \ ,
\end{aligned} \tag{18.2.23}$$

so that estimate (18.2.22), together with the size of the normal form, implies that, for

$|t| \le T_\Lambda$, the motion orthogonal to the fast drift plane is bounded by

$$\begin{aligned} & |I(t) - I(0) - \Pi_{\langle\Lambda\rangle}\big(\mathtt{I}(t) - \mathtt{I}(0)\big)|_2 \\ & \le |I(t) - \mathtt{I}(t)|_2 + |\Pi_{\langle\Lambda\rangle^\perp}\big(\mathtt{I}(t) - \mathtt{I}(0)\big)|_2 + |\mathtt{I}(0) - I(0)|_2 \\ & \le \frac{r_\Lambda}{32\xi} + \frac{r_\Lambda}{4} + \frac{r_\Lambda}{32\xi} \le \frac{3}{4} r_\Lambda \ , \end{aligned} \tag{18.2.24}$$

where we have used the fact that $\xi > 1$. Hence, by (18.2.24), $I(t) \in D^\rho_{\Lambda, r_\Lambda}$ since $I(0) \in D_\Lambda$ and the orbit lies entirely in this set for any $|t| \le T_\Lambda \le \tau_e$; moreover, the definition in (18.1.10) implies

$$I(t) \in \mathbf{D}^\rho_{\Lambda, \frac{3}{4} r_\Lambda}(I(0)) \subset \mathbf{D}^\rho_{\Lambda, r_\Lambda}(I(0)) \ .$$

This fact, together with Lemma 18.1.1, yields

$$|I(t) - I(0)|_2 \le r_j \quad , \qquad \text{where} \qquad r_j \ :\mathrel{\hat{=}} \ \frac{1}{K^{q_j/\alpha_j}} \ , \ \ r_j \ge r_\Lambda \ . \tag{18.2.25}$$

As it is shown in [70] (formula (38)), a careful choice of the constants leads to

$$\max_{j \in \{1, \dots, n-1\}} r_j < \rho(\varepsilon) \ ,$$

which concludes the proof of the first claim of this Lemma.

We now consider the second claim. In this case, for any time t such that $|t| < |\tau_e| < T_\Lambda$ we can repeat the same arguments above and find $I(t) \in \mathbf{D}^\rho_{\Lambda, \frac{3}{4} r_\Lambda}(I(0))$. Then, by construction, the escape time satisfies

$$I(\tau_e) \in \text{closure}\Big(\mathbf{D}^\rho_{\Lambda, \frac{3}{4} r_\Lambda}(I(0))\Big). \tag{18.2.26}$$

Again, by Lemma 18.1.1, this implies $|I(t) - I(0)|_2 < \rho(\varepsilon)$ for any $|t| < \tau_e < T_\Lambda$, so that, since $I(0) \in B_2\big(I(0), \mathtt{R}(\varepsilon) - (j+1)\rho(\varepsilon)\big)$ one has

$$I(\tau_e) \in B_2\big(I(0), \mathtt{R}(\varepsilon) - j\rho(\varepsilon)\big) \ . \tag{18.2.27}$$

Now, we shall prove that $I(\tau_e) \notin Z_\Lambda$. By definition we have $I(\tau_e) \notin D^\rho_{\Lambda, r_\Lambda}$ and, thanks to (18.1.11), this means that there does not exist any action $I^* \in D_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)\big)$ such that $I(\tau_e)$ belongs to its disc $\mathbf{D}^\rho_{\Lambda, r_\Lambda}(I^*)$. Hence, by (18.1.10), $I(\tau_e)$ must satisfy at least one of the three following conditions:

1. $\nexists I^* \in D_\Lambda \cap B_2\big(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)\big) \ : \ I(\tau_e) \in \bigcup_{I' \in I^* + \langle\Lambda\rangle} B_2(I', r_\Lambda)$;

2. $I(\tau_e) \notin Z_\Lambda$;

3. $I(\tau_e) \notin B_2\big(I_0, \mathtt{R}(\varepsilon) - \rho(\varepsilon)\big)$.

By taking (18.2.26) and (18.2.27) into account, we see that the first and the third possibility cannot occur. Therefore, there must exist a maximal lattice $\Lambda' \neq \Lambda$ and a resonant zone $Z_{\Lambda'}$ such that $I(\tau_e) \in Z_{\Lambda'}$. Moreover, Lemma 18.1.3, insures that $\dim \Lambda' \neq \dim \Lambda$ so that $I(\tau_e) \notin Z_j$. The second decomposition in (18.1.8) together with (18.2.27) and (18.1.9) implies that $I(\tau_e)$ belongs to a resonant block of lower multiplicity, hence the claim. □

Remark 18.2.2. The decompositions in (18.1.8) are a covering of $B(I_0, \mathtt{R}(\varepsilon))$ but they are not a partition since, in general, $D_i \cap D_j \neq \emptyset$ for $j > i+1$. Hence, nothing prevents $I(\tau_e)$ from belonging to a resonant block of strictly higher multiplicity than the starting one. If this happens, however, thanks to the construction in (18.1.8), one is insured that $I(\tau_e)$ will also belong to another block associated to a lower order resonance. One therefore chooses the block in which to study the evolution of the actions once they leave the resonant zone they started at. This is at the core of the *resonant trap argument*, which is discussed in the sequel.

Proof of Theorem 14.3.1. Theorem 14.3.1 follows from Lemmas 18.2.1 and 18.2.2. Indeed, for any initial condition in the action variables $I_0 \in B_\infty(0, R/4)$, we consider the ball $B_2(I_0, \mathtt{R}(\varepsilon))$ and the following dichotomy holds:

1. either I_0 belongs to the completely non-resonant domain D_0^ρ, in which case the proof ends here thanks to Lemma 18.2.1;

2. or for some $j \in \{1, ..., n-1\}$ and some maximal $\Lambda \subset \mathbb{Z}_K^n$ of rank j, $I_0 \in D_\Lambda \cap B\big(I_0, \mathtt{R}(\varepsilon) - (j+1)\rho(\varepsilon)\big)$.

In the second case, Lemma 18.2.2 applies and one has another dichotomy:

1. either $|I(t) - I(0)|_2 \lesssim \rho(\varepsilon) :\mathrel{\mathring{=}} \varepsilon^b$ over a time T_Λ; in this case the Theorem is proven since, taking into account the fact that the analyticity width in Lemma 18.2.1 satisfies $r \mathrel{\mathring{=}} \varepsilon^{1/2}$, one has

$$\begin{aligned} \mathbf{T}(\varepsilon) := T_0 :\mathrel{\mathring{=}} & \frac{1}{|(1+a\ell)\ln\varepsilon|^{\ell-1}\ \varepsilon^{a(\ell-1)+1/2}} \mathrel{\mathring{=}} \frac{r}{|(1+a\ell)\ln\varepsilon|^{\ell-1}\ \varepsilon^{a(\ell-1)+1}} \\ \lesssim & \frac{\mathtt{T}_j \times r_\Lambda}{|\ln\varepsilon^{6(1+a\ell)}|^{\ell-1}\varepsilon^{a(\ell-1)+1}} \mathrel{\mathring{=}}: T_\Lambda\ , \end{aligned} \tag{18.2.28}$$

where the last inequality is a consequence of the fact that, by (18.2.7), (18.2.17), one can write

$$r \le r_\Lambda \quad \longleftrightarrow \quad \frac{1}{K^{1+q_1}} \le \frac{1}{|\Lambda| K^{q_j}}$$

and that, since $|\Lambda| \le K^j$, the stricter inequality

$$\frac{1}{K^{1+q_1}} \le \frac{1}{K^{j+q_j}} \quad \longleftrightarrow \quad 1+q_1 \ge j+q_j \quad \longleftrightarrow \quad p_1 \ge p_j\ ,$$

is trivially satisfied by the definition of p_1 and p_j, $j \in \{1, ..., n-1\}$, in (18.1.1) and by the fact that the steepness indices are always greater or equal than one.

2. or the actions enter a resonant block $D_i \cap \left(B\big(I_0, \mathtt{R}(\varepsilon) - j\rho(\varepsilon)\big)\right)$ corresponding to a resonant lattice of dimension $i < j$ after having travelled a distance $\rho(\varepsilon)$ over a time inferior to the time of escape. In this block, the above arguments can be repeated so that, after having possibly visited at most $n-1$ blocks, overall the actions can travel at most a distance $(n-1)\rho(\varepsilon)$ before entering the completely non-resonant block, in which they are trapped for a time T_0 given by Lemma 18.2.1 and they travel for another length $\rho(\varepsilon)$. Thanks to (18.1.9), by construction one has $|I(t) - I(0)| \leq n\rho(\varepsilon) = \frac{1}{2}\mathtt{R}(\varepsilon) \stackrel{\circ}{=} \varepsilon^b$.

This is the so-called *resonant trap argument* and concludes the proof of Theorem 14.3.1, once one sets

$$\mathtt{a} = a(\ell - 1) + \frac{1}{2} \quad , \qquad \mathtt{b} = b\,.$$

□

Part IV

Quantitative Morse-Sard's Theory for nearly-integrable Hamiltonians near simple resonances

Abstract

This part of the thesis is a work in progress. In the sequel we demonstrate a preparatory step which is crucial in order to prove a conjecture by Arnol'd, Kozlov and Neishtadt stating that the complementary set of invariant KAM tori of a generic nearly-integrable system of the kind $H(I,\theta) = h(I) + \varepsilon f(I,\theta)$ has Lebesgue measure $O(\varepsilon)$. The result we prove is a cornerstone which is necessary to extend to the larger class of real-analytic systems $H(I,\theta) = I^2/2 + \varepsilon f(I,\theta)$, and, possibly, even to the complete real-analytic system $H(I,\theta) = h(I) + \varepsilon f(I,\theta)$ a former result by Biasco and Chierchia on the phase space of systems of the form $H(I,\theta) = I^2/2 + \varepsilon f(\theta)$. Biasco and Chierchia's result, in turn, is fundamental in order to prove that the measure of the complementary set of invariant KAM tori for a generic system is bounded from above by $O(\varepsilon)$, which constitutes one block of the aforementioned conjecture. In particular, we will show that a dependence of the perturbation f on the action variables adds non-trivial obstacles to the proof of Biasco and Chierchia's result, which are tackled by making use of tools of quantitative Morse-Sard's Theory which were developed by Yomdin and Yomdin-Comte.

Chapter 19

Heuristic introduction and state of the art

19.1 Preliminary considerations

Let $n \geq 2$ be a positive integer, and let D be an open subset of $\mathbb{R}^n$. We consider those analytic Hamiltonian systems whose associated function H is the sum of an integrable part (in the sense of Arnol'd-Liouville) and of a small perturbation. Namely, indicating the n-dimensional torus by $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, and making use of standard action-angle coordinates $(I,\theta) \in D\times\mathbb{T}^n$ associated to the symplectic two-form $\Omega := \sum_{i=1}^n dI_i \wedge d\theta_i$, we are interested in those systems whose Hamiltonian reads

$$H(I,\theta) := h(I) + \varepsilon f(I,\theta) \quad , \qquad H \in C^\omega(D\times\mathbb{T}^n;\mathbb{R}) \tag{19.1.1}$$

where $\varepsilon > 0$ is a small parameter which characterizes the size of the perturbation f w.r.t. the integrable part h. The equations of motion associated to H are

$$\Omega(X_H,\cdot) = -dH \quad , \qquad X_H := \mathcal{J}(\nabla H)^\dagger \quad , \tag{19.1.2}$$

where $\mathcal{J}$ is the standard symplectic matrix.

In this work we demonstrate a preparatory step which is crucial in order to prove a conjecture by Arnol'd, Kozlov and Neishtadt stating that the complementary set of invariant KAM tori of a generic nearly-integrable system of the kind (19.1.1)-(19.1.2) has Lebesgue measure $O(\varepsilon)$. Namely, the theorem we prove is a cornerstone which is necessary to extend to the larger class of systems

$$H(I,\theta) = \frac{I^2}{2} + \varepsilon f(I,\theta) \quad , \qquad f \in C^\omega(D\times\mathbb{T}^n) \quad , \qquad 0 < \varepsilon << 1$$

and, possibly, even to the complete system (19.1.1), a former result by Biasco and Chierchia (see [24]) on the the phase space of systems of the form

$$H(I,\theta) = \frac{I^2}{2} + \varepsilon f(\theta) \quad , \qquad f \in C^\omega(\mathbb{T}^n) \quad , \qquad 0 < \varepsilon << 1 \, .$$

Biasco and Chierchia's result, in turn, is fundamental in order to prove that the measure of the complementary set of invariant KAM tori for a generic system (19.1.1) is bounded from above by $O(\varepsilon)$, which constitutes one block of the aforementioned conjecture (see [22]). The other part of the conjecture, stating that a generic system (19.1.1) has a set of Lebesgue measure $\geq O(\varepsilon)$ that does not contain any invariant torus, is a very hard problem to tackle and will not be discussed here.

The building block of this work can be stated in the following way

Theorem 19.1.1. *Let $\mathcal{D}$ be a compact domain of $\mathbb{R}^2$ and let $\mathcal{B}$ be the unit ball in the space $C^5(\mathcal{D};\mathbb{R})$ endowed with the standard C^5-norm. There exists a constant $\mathsf{C} = \mathsf{C}(\mathcal{D})$ such that, for any quadruplet of functions (F_1, F_2, F_3, F_4), with $F_i \in \mathcal{B}$ for all $i \in \{1,2,3,4\}$, for any $\eta > 0$ sufficiently small, and for every vector $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ lying outside of a set of measure $O(\eta)$ in $\mathbb{R}^4$, the shifted functions*

$$F_i^\lambda(x) := F_i(x) - \lambda_i \qquad i \in \{1,2,3,4\}$$

verify

$$\min_{x\in\mathcal{D}} \left(|F_1^\lambda(x)| + |F_2^\lambda(x)| + |F_3^\lambda(x)| + |F_4^\lambda(x)|\right) \geq \mathsf{C}\,\eta^{19/6}\,. \tag{19.1.3}$$

As we will see in section 20, the proof of this statement relies on quantitative results of Morse-Sard's Theory developed by Yomdin [114] and Yomdin-Comte [119]. In the same section, the rôle of Theorem 19.1.1 in the proof of the conjecture by Arnol'd-Kozlov and Neishtadt is discussed from a heuristic point of view. In the following paragraphs, instead, we will make an overview of the main ideas lying behind this conjecture, and we will present the state of the art without entering into too many technicalities.

19.2 On the conjecture by Arnold, Kozlov and Neishtadt

From a general point of view, the structure of the phase space of systems governed by (19.1.1)-(19.1.2) is the object of study of classical Kolmogorov-Arnol'd-Moser (KAM) Theory (see e.g. [5]), whose fundamental results for analytic systems are briefly discussed in the sequel.

For any vector $\omega \in \mathbb{R}^n$, we start by giving the following

Definition 19.2.1. System (19.1.1)-(19.1.2) has a primary invariant torus carrying quasi periodic motions of frequency ω if there exists a real-analytic embedding $\Phi : \mathbb{T}^n \longrightarrow D \times \mathbb{T}^n$ verifying

1. the image $\Phi(\mathbb{T}^n)$ is a Lagrangian set which is invariant for the dynamics generated by Hamiltonian (19.1.1);

2. the flow of system (19.1.1)-(19.1.2) restricted to $\Phi(\mathbb{T}^n)$ is conjugated to the linear quasi-periodic flow $q \longmapsto q + \omega t$ in $\mathbb{T}^n$;

3. the image $\Phi(\mathbb{T}^n)$ is a graph over $\mathbb{T}^n$ in $D \times \mathbb{T}^n$.

Then, it is well-known that

1. for $\varepsilon = 0$, system (19.1.1)-(19.1.2) is integrable in the sense of Arnol'd-Liouville, and its phase space is foliated by primary invariant tori carrying quasi-periodic motions;

2. for $\varepsilon > 0$ sufficiently small, if the frequency map $\omega := \nabla h : D \longrightarrow \mathbb{R}$ satisfies the Kolmogorov non-degeneracy condition $\det(\partial^2 h/\partial I^2) \neq 0$ on D (i.e., if it is a local diffeomorphism on D), classical KAM Theory ensures the existence of a set of relative Lebesgue measure $1 - O(\sqrt{\varepsilon})$ of primary invariant tori carrying quasi-periodic motions. In particular, each invariant torus is a deformation of order $O(\sqrt{\varepsilon})$ of an invariant torus of the unperturbed system.

Classical KAM Theory strongly relies on the study of the commensurability conditions (resonances) satisfied by the frequencies $\omega := \dot{h}/\dot{I}$ of the unperturbed system h. In particular, for fixed $\gamma > 0$, $\tau \geq n-1$, the invariant tori of a Kolmogorov non-degenerate integrable system whose associated frequencies ω satisfy the (γ, τ)-Diophantine condition

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbb{Z}^n \backslash \{0\} \quad , \qquad |k| := \sum_{i=1}^n |k_i| \tag{19.2.1}$$

persist under any sufficiently small perturbation.

It is expected that primary invariant tori of a generic Kolmogorov non-degenerate nearly-integrable system do not fill the phase space homogeneously. In order to give a heuristic justification of this fact, we start by providing the two following definitions:

Definition 19.2.2. Let $K > 1$ be a real number, and $d \in \{1, \dots, n\}$. A K-resonance of order d in frequency space is the locus of points satisfying exactly d linearly independent relations of the kind $k^1 \cdot \omega = 0, \dots, k^d \cdot \omega = 0$, where $k^1, \dots, k^d \in \mathbb{Z}^n \backslash \{0\}$ are multi-integers whose lengths $|k^i|$ verify $|k^i| \leq K$, where $i \in \{1, \dots, d\}$.

Definition 19.2.3. Consider two real numbers $K > 1$ and $\alpha_0 > 0$. The (α_0, K)-completely non-resonant block in frequency space is the subset of frequencies ω verifying

$$|k \cdot \omega| \geq \alpha_0 \qquad \forall k \in \mathbb{Z}^n \backslash \{0\} \ , \ \ |k| \leq K \ .$$

One has that - for appropriately chosen values of $K > 1$ and $\alpha_0 > 0$ - the frequency space can be covered by suitably constructed open neighborhoods of K-resonances and by the (α_0, K)-non-resonant block. Provided that the Kolmogorov non-degeneracy assumption holds (which amounts to asking for the local invertibility of the frequency map), such a covering can be pulled back into action space. The construction of a resonant covering adapted to the considered problem is, in general, a very delicate matter, and there is no unique choice of the parameters determining the sizes of the resonant neighborhoods (see e.g. [104], [70]). Here, we follow Biasco and Chierchia's construction in ref. [24] (Proposition 2.1) and we admit the existence of a covering of the action space D into three blocks

$$D = D_0 \bigcup D_1 \bigcup D_2 \tag{19.2.2}$$

having, roughly speaking[1], the following properties

- D_0 is (α_0, K) completely non-resonant for a suitable value α_0. Moreover, there exists $c > 0$ such that D_0 is filled up to a set of Lebesgue measure $O(e^{-1/\varepsilon^c})$ by primary invariant tori carrying quasi-periodic motions associated to Diophantine frequencies.

- D_1 is the inverse image in action space of neighborhoods of size $O(\sqrt{\varepsilon})$ around hyperplanes associated to K-resonances of order 1 (simple resonances determined by exactly one linear relation $k \cdot \omega = 0$, with $|k| \leq K$). If one chooses $K \sim |\ln \varepsilon|^a$, with $a > 0$, up to logarithmic corrections the measure of D_1 is of order $O(\sqrt{\varepsilon})$.

- D_2 contains K-resonances of order higher or equal than two and, up to logarithmic corrections, has measure $O(\varepsilon)$ for $K \sim |\ln \varepsilon|^a$, $a > 0$.

Remark 19.2.1. As it will be briefly mentioned in the next section, in Biasco and Chierchia's construction one actually needs to set two "ultra-violet" cut-offs K_1 and K_2, with $K_2 > K_1$. K_1 plays the same rôle of K in the heuristics above, whereas K_2 is introduced as one needs - for technical reasons which will not be discussed here - to eliminate neighborhoods of K_2-resonances of order $d \geq 2$ from the block D_1. However, for the purpose of this section, we will not need this distinction and we will only consider the cut-off $K_1 = K$.

Heuristically speaking, it is natural to expect that most of the primary tori of system (19.1.1)-(19.1.2) are to be found in D_0. Infact, as it was conjectured by Arnol'd, Kozlov and Neishtadt in [5], D_2 is not expected to contain a large set of invariant tori, as its dynamics is essentially non perturbative. Namely, one has the following

Conjecture (Arnol'd, Kozlov, Neishtadt (ref. [5], Remark 6.8, p.285)

It is natural to expect that in a generic system with three or more degrees of freedom the measure of the 'non-torus' set has order ε. Indeed, the $O(\sqrt{\varepsilon})$-neighbourhoods of two resonant surfaces intersect in a domain of measure $\sim \varepsilon$. In this domain, after the partial averaging taking into account the resonances under consideration, normalizing the deviations of the "actions" from the resonant values by the quantity $\sqrt{\varepsilon}$, normalizing time, and discarding the terms of higher order, we obtain a Hamiltonian of the form

[1] The estimtes presented below hold up to logarithmic corrections.

$\frac{1}{2}\langle Ap, p\rangle + V(q_1, q_2)$*, which does not involve a small parameter (...). Generally speaking, for this Hamiltonian there is a set of measure* ~ 1 *that does not contain points of invariant tori. Returning to the original variables we obtain a 'non-torus' set of measure* $\sim \varepsilon$.

On the other hand, the example of the classical pendulum governed by the Hamiltonian function $I^2/2 + \varepsilon\cos(\theta)$, with $(I, \theta) \in \mathbb{R} \times \mathbb{T}$, shows that the measure of the complementary set of primary KAM tori can attain a value of order $O(\sqrt{\varepsilon})$. Infact, the invariant sets which are inside the separatrices of the pendulum are all homotopically trivial and thus cannot be primary KAM tori (as they cannot be graphs over $\mathbb{T}$). On the other hand, the region outside of the separatrices is filled by primary KAM tori which are graphs over $\mathbb{T}$. Hence, as the area of the region inside the separatrices is $O(\sqrt{\varepsilon})$, this is also the size of the complementary set of invariant primary tori for the pendulum.

Therefore, by the above heuristic arguments one expects most of KAM primary tori for a generic system to be in D_0 and their complementary set to have measure bounded by $O(\sqrt{\varepsilon})$ (possibly, up to logarithmic corrections).

However, as the example of the classic pendulum shows, primary invariant tori are not the only type of tori that a nearly-integrable system may possess. Indeed, the region inside the separatrices of the pendulum is filled by homotopically trivial invariant sets that are images of embeddings of $\mathbb{T}^n$, and on which the dynamics is periodic. These sets do not appear in the non-perturbative regime associated to $\varepsilon = 0$ and are to be considered as a pure byproduct of the perturbation. We generalize this kind of sets by giving the following

Definition 19.2.4. System (19.1.1) -(19.1.2) has a secondary invariant torus carrying quasi periodic motions of frequency ω if there exists a real-analytic embedding $\Phi : \mathbb{T}^n \longrightarrow D \times \mathbb{T}^n$ verifying

1. the image $\Phi(\mathbb{T}^n)$ is invariant for the dynamics generated by Hamiltonian (19.1.1);
2. the flow of system (19.1.1)-(19.1.2) restricted to $\Phi(\mathbb{T}^n)$ is conjugated to the linear quasi-periodic flow $q \longmapsto q + \omega t$ in $\mathbb{T}^n$;
3. for fixed angles $(\overline{q}_2, \ldots, \overline{q}_n) \in \mathbb{T}^{n-1}$, the image $\Phi(q_1, \overline{q}_2, \ldots, \overline{q}_n)$ is homotopically trivial in $D \times \mathbb{T}^1$.

In general, we do not expect secondary tori to fill D_2 for, as we have already pointed out, the dynamics in that region is non-perturbative. However, secondary tori are expected to appear in D_1. To understand this from a heuristic point of view, we firstly remind that any frequency in D_1 lies in a neighborhood of some resonant hyperplane of the kind $k \cdot \omega = 0$, with $k \in \mathbb{Z}^n$, $|k| \leq K$, and no other linearly independent resonant relations are verified in such a neighborhood. Therefore, one can apply perturbation theory near a given simple resonance in D_1 and put Hamiltonian H into resonant normal form (see e.g. [104]) by averaging out $n-1$ angles.

Hence, up to a small remainder, near each simple resonance of D_1 system (19.1.1)-(19.1.2) is symplectically conjugated to a system depending on only one angle (therefore, integrable). Then, one may build action-angle coordinates (J,φ) for this system and indicate by $h^\star(J)$ the associated integrable Hamiltonian function. Taking into account the small remainder due to the normal form - indicated by $f^\star$ - one finally obtains that system (19.1.1)-(19.1.2) can be symplectically conjugated to a system $H^\star(J,\varphi) = h^\star(J) + \varepsilon f^\star(J,\varphi)$ near simple resonances in D_1. If the Kolmogorov non-degeneracy condition is satisfied by $h^\star(J)$, then KAM Theorem can be applied, and the region of D_1 near the simple resonance that one is considering is filled with primary Lagrangian tori of $H^\star(J,\varphi)$, which correspond to secondary tori of the initial system $H(I,\theta)$ (see [23], [24]).

Moreover, indicating by s the analyticity width of H, by standard result of perturbation theory the size of the remainder $f^\star$ is of order $O(e^{-Ks})$ (see [104]), where K is the usual "ultraviolet cut-off" on the length of the vector generating the simple resonance. If one chooses, for example, $K \sim |\ln\varepsilon|^a$ for some $a > 1$, then the remainder is of order $O(\varepsilon' := e^{-|\ln\varepsilon|^a s})$, and the classical KAM estimate yields that the measure of the complementary set of secondary tori near the considered simple resonance (that is, of primary tori of system $H^\star$) is bounded by $O(\sqrt{\varepsilon'} := \sqrt{e^{-|\ln\varepsilon|^a s}})$, which is asymptotically smaller than $O(\varepsilon)$ for $a > 1$. Summing up all possible simple resonances of length $1 \le |k| \le K = K(\varepsilon) \sim |\ln\varepsilon|^a$ one has that, up to logarithmic corrections, D_1 is filled by secondary tori up to a set of measure bounded by $O(\sqrt{e^{-|\ln\varepsilon|^a s}})$.

Hence, if one manages to make the heuristic reasoning above rigorous for a generic system (19.1.1) and takes secondary tori into account, the measure of the set of "non-tori" in phase space is bounded by the measure of D_2, that is - up to logarithmic corrections - by $O(\varepsilon)$, as stated by the conjecture of Arnol'd, Kozlov, and Neishtadt.

Many technical difficulties arise when trying to demonstrate the heuristics above rigorously. Up to now, the only known result is the one of, Biasco and Chierchia (see [22], [24], [23]) who have carried out a demonstration of the conjecture by Arnold, Kozlov and Neishtadt for "natural" mechanical systems of the kind

$$H(I,\theta) = \frac{I^2}{2} + \varepsilon f(\theta) \quad , \qquad f \in C^\omega(\mathbb{T}^n)\,. \tag{19.2.3}$$

In the present work the goal is to present a result which constitutes a first step in order to prove the conjecture for the more general class of systems

$$H(I,\theta) = \frac{I^2}{2} + \varepsilon f(I,\theta) \quad , \qquad f \in C^\omega(D \times \mathbb{T}^n) \tag{19.2.4}$$

and, possibly, even to

$$H(I,\theta) = h(I) + \varepsilon f(I,\theta) \quad , \qquad H \in C^\omega(D \times \mathbb{T}^n)\,. \tag{19.2.5}$$

In the next two subsections, we will make on overview about Biasco and Chierchia's strategy, and we will show how it must be modified in case the perturbation depends on the action variables. In particular, we will see that the latter case adds a non-trivial difficulty to the problem, which is tackled by making use of quantitative tools of Morse-Sard's Theory.

19.3 Heuristics on KAM Theory for secondary tori.

Biasco and Chierchia's proof of the conjecture is splitted into two main parts, which are discussed below from a heuristic point of view.

Conjugation to a "pendulum-like" system close to simple resonances (see [24])

Given a multi-integer $k = (k_1, \dots, k_n) \in \mathbb{Z}^n \backslash \{0\}$ satisfying

$$k_1 > 0 \quad , \qquad \gcd(k_1, \dots, k_n) = 1 \quad , \qquad |k| \le K \; , \tag{19.3.1}$$

we indicate by $D_1^k \subset D_1$ its associated simple resonant zone, that is the neighborhood of size $O(\sqrt{\varepsilon})$ around the resonant manifold

$$\mathcal{M}_1^k := \{ I \in D_1 \,|\, k \cdot \omega(I) = 0 \} \; .$$

We stress that, for $I \in \mathcal{M}_1^k$, the frequency $\omega(I)$ does not satisfy any other linearly independent equation of the kind $k' \cdot \omega(I) = 0$ for any $k' \in \mathbb{Z}^n \backslash \{0\}$, $|k'| \le K$.

The idea, here, is to show that close to simple resonances generated by a sufficiently large k verifying (19.3.1), any generic, analytic, nearly-integrable system behaves like a pendulum. The resonances associated to generating vectors k of small size are treated separately, as we will briefly discuss in the sequel.

By classical results of perturbation theory (see e.g. the normal form in [104]), there exists an analytic symplectic change of coordinates Φ^k defined in a complex neighborhood of $D_1^k \times \mathbb{T}^n$ that conjugates the analytic system[2] $H(I,\theta) = h(I) + \varepsilon f(\theta)$ to

$$H \circ \Phi^k(I,\theta) = h^k(I) + \varepsilon \left(g^k(I, k \cdot \theta) + f_\star^k(I,\theta) \right) \; , \tag{19.3.2}$$

where g^k depends on the sole angle $\varphi = k \cdot \theta$ and $f_\star^k$ is a suitably small remainder. It is known that $g^k(I, k \cdot \theta)$ is "close" (in a sense to be defined rigorously) to the "resonant part of the perturbation" determined by the Fourier expansion $\sum_{j \in \mathbb{Z}} f_{jk} e^{i jk \cdot \theta}$, which contains only those harmonics that are multiple of the generating vector k. Namely, if one supposes $f_k \neq 0$, and takes into account the fact that - for a suitable functional norm $|| \cdot ||$ - the Fourier coefficients decay as $|f_k| \simeq O(||f|| e^{-|k|s})$, where s denotes

[2] We observe that, in this part of the proof, no assumption is made about the analytic integrable part $h(I)$, whose form is not necessarily $I^2/2$.

the complex analyticity width of $H(I,\theta)$, one can write

$$\begin{aligned} & g^k(I,k\cdot\theta) \\ & = \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk}e^{ijk\cdot\theta} + g^k(I,k\cdot\theta) - \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk}e^{ijk\cdot\theta} \\ & \simeq f_k e^{ik\cdot\theta} + \overline{f}_k e^{-ik\cdot\theta} + O(||f||e^{-2|k|s}) + g^k(I,k\cdot\theta) - \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk}e^{ijk\cdot\theta} \\ & = 2|f_k|\left(\cos(k\cdot x+\theta^k) + O(e^{-|k|s}) + \frac{g^k(I,k\cdot\theta) - \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk}e^{ijk\cdot\theta}}{|f_k|}\right) . \end{aligned} \tag{19.3.3}$$

Therefore, from heuristic point of view, if one compares (19.3.2) with (19.3.3), takes into account the fact that $|g^k(I,k\cdot\theta)-\sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk}e^{ijk\cdot\theta}| \simeq o(1)$, and assumes $1/s << |k| \leq K$, then one can write

$$H\circ\Phi^k(I) =: h^k(I) + 2\varepsilon|f_k|\left(\cos(k\cdot\theta+\theta^k) + N^k(I,\theta)\right) , \quad 1/s << |k| \leq K \tag{19.3.4}$$

where $N^k(I,\theta)$ is a small remainder. By these heuristic arguments, it is clear that if $h(I) = I^2/2$, then the phase portrait of system (19.3.4) is close to that of the standard pendulum $I^2/2 + 2\varepsilon|f_k|\cos(k\cdot\theta+\theta^k)$.

In [24] the authors show that the heuristics above can be made rigorous. In particular, under the hypotheses that

1. ε is sufficiently small;
2. $1/s << |k| \leq K$;
3. for some $\delta > 0$, there exists $\tau(\delta) > 0$ such that, for $|k| > \tau(\delta)$, one has

$$|f_k| \geq \delta\frac{e^{-|k|s}}{|k|^n} , \tag{19.3.5}$$

then there exists an analytic symplectic change of coordinates Ψ^k defined in a complex neighborhood of $D_1^k \times \mathbb{T}^n$ that conjugates system (19.2.3) to

$$H\circ\Psi^k(I) =: h^k(I) + 2|f_k|\varepsilon\left(\cos(k\cdot x+\theta^k) + G^k(I,k\cdot\theta) + R^k(I,\theta)\right) , \tag{19.3.6}$$

where the sizes of G^k and R^k are suitably small so that $\cos(k\cdot x+\theta^k)+G^k(I,k\cdot\theta)$ is a Morse function with only one minimum and one maximum, and system (19.3.6) is dynamically equivalent to a simple pendulum.

Moreover, hypothesis (19.3.5) is proved to be generic - in measure and topological sense - in the space $C^\omega(\mathbb{T}^n)$ (endowed with a suitable norm).

The above result is particularly important, because it shows that close to simple resonances generated by a vector $k \in \mathbb{Z}^n\setminus\{0\}$ satisfying (19.3.1) and $1/s << |k| \leq K$, the behavior of a generic analytic system of the kind $H(I,\theta) = h(I) + \varepsilon f(\theta)$ is dynamically equivalent to that of a pendulum. As we will discuss in the next subparagraph, this proves very useful when dealing with the construction of action-angle coordinates for the averaged system near simple resonances.

From a technical point of view, two major technical obstacles are to be overcome when passing from the heuristics to the rigorous proof. Firstly, a direct application of classical perturbation theory (e.g. the standard normal form considered in [104]) proves insufficient to make the remainder in N^k in (19.3.4) small enough. Hence, a more refined normal form, characterized by a "small" consumption of the analyticity width s must be introduced. Secondly, a suitable resonant patchwork must be constructed in order to avoid the possibility that $|f_k|$ and the remainder R^k in (19.3.6) have comparable sizes.

Construction of action-angle coordinates for the averaged system near simple resonances and verification of Kolmogorov's non-degeneracy.

In this step, only systems of the form (19.2.3), where the integrable part of the Hamiltonian has the form of a kinetic energy, are considered. The second part of Biasco and Chierchia's proof consists in constructing action-angle coordinates for the averaged system (19.3.2) near simple resonances and in checking that Kolmogorov's non-degeneracy condition is satisfied. In particular, simple resonances associated to the "low modes" $k \in \mathbb{Z}^n$, $1 \leq |k| \lesssim 1/s$ are studied separately from those generated by the high modes $k \in \mathbb{Z}^n$, $1/s << |k| \leq K$.

Low modes. The phase portraits of this type of resonances can be very complicated; generally speaking, a finite number of islands of stability delimited by separatrices appear among free motions associated to homotopically non-trivial orbits (see Figure 19.1). Moreover, it is also possible that islands are nested inside each other. However, for a given system, by construction the number of resonances associated to the low modes is constant and independent of the value of K. Hence, in order to treat the low modes one has in principle to construct action-angle variables and to check Kolmogorov's non-degeneracy for a finite number of complicated systems. In particular, verifying that Kolmogorov's determinant is not zero is a highly non-trivial task which requires the application of technical non-perturbative arguments. We will not treat this aspect here, as this would take the discussion far from the main focus of the present work.

High modes As the ultraviolet cut-off K can be arbitrarily large (usually, one chooses $K(\varepsilon) \longrightarrow +\infty$ when $\varepsilon \longrightarrow 0$), the number of resonances associated to high modes which must be considered is - in principle - unbounded when $\varepsilon \longrightarrow 0$. Without any additional information, this would be a serious obstacle, since one would have to control an infinite number of different systems, each with different parameters that might degener-

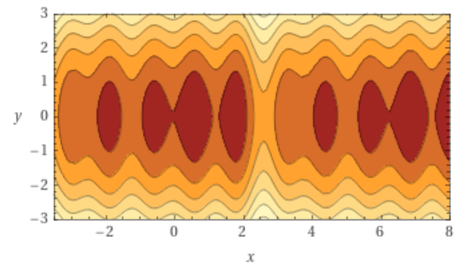

Figure 19.1: Phase portrait of system $H(y,x) := \frac{y^2}{2} - \frac{1}{2}\cos(x) + \frac{2}{5}\cos(2x+1) + \frac{3}{10}\cos(3x-2) - \frac{1}{5}\cos(4x-1) + \frac{1}{2}\cos(5x)$, taken as an example of behavior near a "low mode" resonance. We see that the topology of this system is very complicated: many separatrices lie between nested islands of stability and free librations.

ate. However, as it was discussed heuristically in the previous subsection, for a generic Hamiltonian the phase portrait of any high mode resonance is topologically equivalent to that of the simple pendulum, that is a Morse function with only one maximum and one minimum. This important property simplifies dramatically the construction of the action-angle coordinates and the verification of the Kolmogorov's non-degeneracy, since it solves the problem of having a uniform control on the parameters.

Chapter 20

Main result

In this section, we present a result which is meant to extend the first step of Biasco and Chierchia's construction to generic, nearly-integrable Hamiltonian systems of the form

$$H(I,\theta) = h(I) + \varepsilon f(I,\theta) \quad , \qquad H \in C^{\omega}(D \times \mathbb{T}^n)\,, \tag{20.0.1}$$

where D is a domain of the euclidean space $\mathbb{R}^n$, and $\mathbb{T}^n := \mathbb{R}^n \backslash \mathbb{Z}^n$ is the standard torus.

20.1 Heuristic motivation and strategy

As in the case of a perturbation depending only on the angular variables, simple resonances associated to low modes are considered separately. Hence, in the sequel, we will concentrate only on resonances associated to high modes, generated by vectors $k = (k_1, \dots, k_n) \in \mathbb{Z}^n \backslash \{0\}$ verifying

$$k_1 > 0 \quad , \qquad \gcd(k_1, \dots, k_n) = 1 \quad , \qquad 1/s << |k| \leq K\,, \tag{20.1.1}$$

where $s > 0$ is the analyticity width of Hamiltonian H.

Around each resonance of this type, the symplectic conjugation to a Morse function with only one maximum and one minimum described in subsection 19.3 fails for a generic system governed by Hamiltonian (20.0.1). Indeed, in subsection 19.3 we considered $f \in C^{\omega}(\mathbb{T}^n)$, so that the coefficients f_k of the Fourier expansion $\sum_{k \in \mathbb{Z}^n} f_k e^{i\,k\cdot\theta}$ were constants; under this condition - as we have already mentioned in the introduction - one can prove that the crucial condition (19.3.5) is generically satisfied. However, in the case of system (20.0.1), the Fourier coefficients $f_k(I)$ are holomorphic functions, and it is false that a generic holomorphic function does not have any zeroes on its domain. Therefore, condition (19.3.5) cannot be expected to be satisfied by a generic perturbation $f \in C^{\omega}(D \times \mathbb{T}^n)$.

In order to overcome this difficulty, the idea is to firstly apply the aforementioned Biasco and Chierchia's improved Normal Form (see [24], Prop. 4.1) to system (20.0.1) inside the domain $D_1^k \subset D_1$ associated to a simple resonance of generating vector k verifying (20.1.1). Without entering into too many technicalities, the authors show that - if ε is sufficiently small - then there exists a real-analytic, symplectic change of coordinates Ψ_k defined in a complex neighborhood of D_1^k which conjugates system (20.0.1) to

$$H \circ \Psi^k(I) = h^k(I) + \varepsilon \sum_{j \in \mathbb{Z}\setminus\{0\}} f_{jk}(I) e^{i\,jk\cdot\theta} + \varepsilon \mathsf{R}^k(I,\theta)\,, \tag{20.1.2}$$

where R^k is a suitably small remainder in a sense which will be specified in the sequel. Secondly, depending on the number of degrees of freedom n of the considered system, the strategy consists in fixing a suitable positive integer $\ell = \ell(n) \geq 2$ and in taking system

$$h^k(I)+\varepsilon \sum_{j\in\mathbb{Z}\setminus\{0\}, |j|\leq \ell} f_{jk}(I)\, e^{i\,jk\cdot\theta} = h^k(I)+2\varepsilon \sum_{j\in\mathbb{Z}\setminus\{0\}, |j|\leq \ell} |f_{jk}(I)| \cos(j(k\cdot\theta)+\theta^{jk}) \tag{20.1.3}$$

to be the integrable approximation near the simple resonance, instead of choosing

$$h^k(I) + \varepsilon f_k(I) e^{i\,k\cdot\theta} + \varepsilon \overline{f}_k(I) e^{-i\,k\cdot\theta} = h^k(I) + 2|f_k(I)| \cos(k\cdot x + \theta^k) \tag{20.1.4}$$

as in the original framework (see (19.3.3)). The rest of this section is devoted to a heuristic discussion about how the choice of system (20.1.3) instead of (20.1.4) overcomes the difficulties encountered by the original strategy.

Expression (20.1.2) can be rewritten as

$$\begin{aligned} H\circ\Psi^k(I) =& h^k(I) \\ &+ \varepsilon \left(\sum_{j\in\mathbb{Z}\setminus\{0\}, |j|\leq\ell} f_{jk}(I)\, e^{i\,jk\cdot\theta} + \sum_{j\in\mathbb{Z}\setminus\{0\}, |j|>\ell} f_{jk}(I)\, e^{i\,jk\cdot\theta} + \mathsf{R}^k(I,\theta) \right) \end{aligned} \tag{20.1.5}$$

For the above construction to make sense, the term

$$\sum_{j\in\mathbb{Z}\setminus\{0\}, |j|>\ell} f_{jk}(I)\, e^{i\,jk\cdot\theta} + \mathsf{R}^k(I,\theta) = 2 \sum_{\substack{j\in\mathbb{Z}\setminus\{0\} \\ |j|>\ell}} |f_{jk}(I)| \cos(j(k\cdot\theta) + \theta^{jk}) + \mathsf{R}^k(I,\theta) \tag{20.1.6}$$

must be "small" (in a sense to be specified in the sequel) w.r.t. the term

$$\sum_{j\in\mathbb{Z}\setminus\{0\}, |j|\leq\ell} f_{jk}(I)\, e^{i\,jk\cdot\theta} = 2 \sum_{\substack{j\in\mathbb{Z}\setminus\{0\} \\ |j|\leq\ell}} |f_{jk}(I)| \cos(j(k\cdot\theta) + \theta^{jk})\,, \tag{20.1.7}$$

for $1/s \lesssim |k| \leq K$. In order to see under which conditions this can be ensured, we indicate by r the analyticity width of Hamiltonian (20.0.1) w.r.t. the actions, and we fix two numbers $\rho \in]0, r[$, $\sigma \in]0, s[$, to which we associate the complex domains

$$D_\rho := \{ I \in \mathbb{C}^n \,|\, \exists I_0 \in D \text{ satisfying } ||I - I_0|| < \rho \}$$

$$\mathbb{T}^n_\sigma := \left\{ \theta \in \mathbb{T}^n_{\mathbb{C}} \mid \max_{j\in\{1,\ldots,n\}} |\mathrm{Im}\,(\theta_j)| < \sigma \right\} \quad , \qquad \text{where} \quad \mathbb{T}^n_{\mathbb{C}} := \mathbb{C}^n/\mathbb{Z}^n \,.$$

Then, we endow the space $C^\omega(D\times\mathbb{T}^n)$ with the weighted Fourier norm

$$||f||_{\rho,\sigma} := \sup_{I\in D_\rho}\left(\sum_{k\in\mathbb{Z}^n} |f_k(I)| e^{|k|(s-\sigma)}\right) = \sup_{I\in D_\rho}\left(\sum_{k\in\mathbb{Z}^n} |\widetilde{f}_k(I)| e^{-|k|\sigma}\right) ,$$

where we have introduced the weighted Fourier coefficients

$$\widetilde{f}_k(I) := f_k(I)\, e^{|k|s} \qquad \forall k \in \mathbb{Z}^n\backslash\{0\} \,. \tag{20.1.8}$$

At this point, on the one hand we observe that if there exists $\eta' > 0$ such that

$$\min_{I\in D}\left(\sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell} |\widetilde{f}_{jk}(I)|\right) \ge \eta' \tag{20.1.9}$$

then one has the following estimate:

$$\min_{I\in D}\left(\sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell} |\widetilde{f}_k(I)| e^{-|jk|\sigma}\right) \ge \eta'\, e^{-\ell|k|\sigma} \,. \tag{20.1.10}$$

On the other hand, one has

$$\sup_{I\in D_\sigma} \sum_{j\in\mathbb{Z}\backslash\{0\},|j|>\ell} |\widetilde{f}_{jk}(I)|\, e^{-|jk|\sigma} \lesssim O(e^{-(\ell+1)|k|\sigma}) \,. \tag{20.1.11}$$

As for the remainder $\mathsf{R}^k(I,\theta)$, it can be shown (see [24], Th. 2.1 and Prop. 4.1 for technical details) that it satisfies

$$||\mathsf{R}^k(I,\theta)||_{\rho,\sigma} \lesssim O(e^{-cK_2\sigma}) \tag{20.1.12}$$

for a suitable integer $K_2 > (\ell+1)K_1$ and for some uniform constant $c > 0$, provided that one has carefully constructed the resonant covering of the phase space (see Remark 19.2.1).

Therefore, if the crucial condition (20.1.9) is satisfied, relations (20.1.10), (20.1.11) and (20.1.12) together imply that for any $k \in \mathbb{Z}^n$ satisfying

$$k_1 > 0 \quad , \qquad \gcd(k_1,\ldots,k_n) = 1 \quad , \qquad 1/\sigma << |k| \le K \,, \tag{20.1.13}$$

one has

$$\min_{I\in D} \sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell} |\widetilde{f}_{jk}(I)|\, e^{-|jk|\sigma} >> \sup_{I\in D_\rho} \sum_{j\in\mathbb{Z}\backslash\{0\},|j|>\ell} |\widetilde{f}_{jk}(I)|\, e^{-|jk|\sigma} + ||\mathsf{R}^k(I,\theta)||_{\rho,\sigma} \,, \tag{20.1.14}$$

which, together with the obvious estimate

$$\sup_{I\in D_\rho}\left(\sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell}|\widetilde{f}_{jk}(I)|\,e^{-|jk|\sigma}\right)\ge\min_{I\in D}\left(\sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell}|\widetilde{f}_{k}(I)|e^{-|jk|\sigma}\right),$$

implies

$$\left\|\sum_{j\in\mathbb{Z}\backslash\{0\},|j|\le\ell}f_{jk}(I)\,e^{i\,jk\cdot\theta}\right\|_{\rho,\sigma}>>\left\|\sum_{j\in\mathbb{Z}\backslash\{0\},|j|>\ell}f_{jk}(I)\,e^{i\,jk\cdot\theta}\right\|_{\rho,\sigma}+||\mathsf{R}^k(I,\theta)||_{\rho,\sigma}\,. \tag{20.1.15}$$

Estimates (20.1.14)-(20.1.15), translate quantitatively the initial request for (20.1.6) to be small w.r.t. (20.1.7).

Heuristically speaking, we stress the fact that - unlike (19.3.5) - condition (20.1.9) is generic among analytic functions if ℓ is chosen to be sufficiently high. Infact, (20.1.9) is not satisfied for any $\eta>0$ iff the functions $\widetilde{f}_{jk}(I)$, with $j\in\mathbb{Z}^n\backslash\{0\}$, $|j|\le\ell$, have a zero in common, which is not a generic property for large ℓ. Namely, for any given $\ell\ge 2$ and $k\in\mathbb{Z}^n$ satisfying (20.1.13), the functions $f_{jk}(I)$, with $|j|\le\ell$ have a zero in common if and only if

$$\begin{cases}\text{Re}\,\widetilde{f}_k(I_1,\dots,I_n)=0\quad, & \text{Im}\,\widetilde{f}_k(I_1,\dots,I_n)=0\\ \text{Re}\,\widetilde{f}_{2k}(I_1,\dots,I_n)=0\quad, & \text{Im}\,\widetilde{f}_{2k}(I_1,\dots,I_n)=0\\ \dots & \\ \text{Re}\,f_{\ell k}(I_1,\dots,I_n)=0\quad, & \text{Im}\,\widetilde{f}_{\ell k}(I_1,\dots,I_n)=0\,.\end{cases} \tag{20.1.16}$$

The above system contains 2ℓ equations in n real unknowns and has generically no solution if $\ell\ge\lfloor n/2\rfloor+1$ (see e.g. [55], Sect. 3.7-3.8).

However, in order to make the above strategy rigorous and prove the genericity of condition (20.1.9), we need quantitative estimates on how "far from zero" the equations in system (20.1.16) are. The main result of this work , which is stated in the next subsection, answers to this question by making use of the quantitative Morse-Sard's Theory developed by Yomdin in [114] and by Yomdin and Comte in [119]. For the sake of simplicity, in the sequel we only consider the case of functions depending on $n=2$ variables, but - as it will be discussed further - the reasonings can be easily extended to an arbitrary number n of degrees of freedom.

20.2 Main result

Let $\mathcal{D}$ be a compact domain of $\mathbb{R}^2$. We consider the space $C^5(\mathcal{D};\mathbb{R})$ endowed with the standard $||\cdot||_{C^5(\mathcal{D})}$ norm, and we denote by $\mathcal{B}$ the unit ball in this space.

The rest of this section is devoted to the demonstration of the following result:

Theorem 20.2.1. *There exist two constants* $\mathtt{C}_1 = \mathtt{C}_1(\mathcal{D})$ *and* $\mathtt{C}_2 = \mathtt{C}_2(\mathcal{D})$*, and a threshold* $\eta_0 = \eta_0(\mathcal{D})$ *such that, for any quadruplet of functions* (F_1, F_2, F_3, F_4)*, with* $F_i \in \mathcal{B}$ *for all* $i \in \{1,2,3,4\}$*, and for any* $\eta \in (0, \eta_0)$ *there exist two bad sets* $\mathcal{V}_\eta \subset \mathbb{R}^2, \mathcal{E}_\eta \subset \mathbb{R}^2$ *satisfying*

$$\text{meas } \mathcal{V}_\eta \le \mathtt{C}_1 \eta \quad , \qquad \text{meas } \mathcal{E}_\eta \le \mathtt{C}_1 \eta \tag{20.2.1}$$

such that the shifted functions

$$F_i^\lambda(x) := F_i(x) - \lambda_i \quad i \in \{1,2,3,4\} \qquad \lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathbb{R}^2 \backslash \mathcal{V}_\eta) \times (\mathbb{R}^2 \backslash \mathcal{E}_\eta)$$

verify

$$\min_{x \in \mathcal{D}} \left(|F_1^\lambda(x)| + |F_2^\lambda(x)| + |F_3^\lambda(x)| + |F_4^\lambda(x)| \right) \ge \mathtt{C}_2 \, \eta^{19/6} . \tag{20.2.2}$$

Remark 20.2.1. Theorem 20.2.1 puts the heuristics of the previous subsection into a rigorous framework. Namely, if one takes $n = \ell = 2$ and sets

$$F_1 := \text{Re}\, \widetilde{f}_k \, , \quad F_2 := \text{Im}\, \widetilde{f}_k \, , \quad F_3 := \text{Re}\, \widetilde{f}_{2k} \, , \quad F_4 := \text{Im}\, \widetilde{f}_{2k} \, , \tag{20.2.3}$$

then one has that by the equivalence of norms formula (20.2.2) is equivalent to estimate (20.1.9) (up to setting $\eta' := \eta^{19/6}$). The strategy to follow in order to extend Theorem 20.2.1 to an arbitrary number n of degrees of freedom will be discussed in the sext sections.

Remark 20.2.2. Theorem 20.2.1 states a form of genericity for functions that verify condition (20.2.2) for some sufficiently small $\eta > 0$. Infact, by (20.2.1) one has

$$\text{meas} \left(\bigcap_{\eta \in (0,\eta_0)} \mathcal{V}_\eta \right) = \text{meas} \left(\bigcap_{\eta \in (0,\eta_0)} \mathcal{E}_\eta \right) = 0 \, ,$$

so that, for given functions $F_1, F_2, F_3, F_4 \in \mathcal{B}$, for almost every choice of the shifting parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, the shifted functions $F_i^\lambda := F_i - \lambda_i$, $i \in \{1,2,3,4\}$ verify relation (20.2.2) for some $\eta > 0$.

Remark 20.2.3. Theorem 19.1.1 follows easily from Theorem 20.2.1 and from remark 20.2.2.

Theorem 20.2.1 is proven by combining a quantitative result of Morse-Sard's theory together with a quantitative version of the local inversion Theorem (see Appendix I). Namely, in the proof we will make use of the following

Lemma 20.2.1 (Quantitative Morse-Sard's theory, [114], [119]).

For any $m, n \in \mathbb{N}^\star$*,* $m \le n$*, for any* $\delta \in (0,1)$*,* $\mathcal{M} > 0$*, and for any* $g \in C^{2n+1}(\mathcal{D}, \mathbb{R}^m)$ *with* $0 < ||g||_{C^{2n+1}(\mathcal{D})} \le \mathcal{M}$*, there exists a subset* $\mathcal{U}_\delta \in \mathbb{R}^m$ *and a constant* $\mathtt{C} := \mathtt{C}(n, m, \mathcal{M}, \mathcal{D})$ *such that*

$$\text{meas } \mathcal{U}_\delta \le \mathtt{C}\sqrt{\delta} \tag{20.2.4}$$

and, for any $\lambda \in \mathbb{R}^m \backslash \mathcal{U}_\delta$, the function $g^\lambda(z) := g(z) - \lambda$ satisfies at any point $x \in \mathcal{D}$ at least one of the two conditions

$$||g^\lambda(x)|| > \delta \qquad or \qquad ||Dg^\lambda(x)\xi|| > \delta^{\frac{n}{n+1}}||\xi|| \;\forall \xi \in \mathbb{R}^n \;, \tag{20.2.5}$$

where $||\cdot||$ indicates the standard euclidean norm.

The above Lemma is proven by using the quantitative Morse-Sard's theory developed by Yomdin and Comte in [114] and [119] (see also Appendix I, where the basic concepts of Morse function and Sard's Theorem are recalled). The given statement can be found in [99] [1], where it is exploited in order to prove the prevalence of a class of functions (the so-called "Diophantine-steep" functions) which plays an important rôle in the study of the effective stability of nearly-integrable Hamiltonian systems.

20.2.1 Proof of Theorem 20.2.1

In the formulas which will appear henceforth, we will often make use of the symbols $\lessdot$, $\gtrdot$ and $\mathrel{\hat{=}}$ to indicate the presence of constants depending at most on the form of the domain $\mathcal{D} \subset \mathbb{R}^2$.

We assume the setting of Theorem 20.2.1 with $m = n = 2$ and - as we will work with functions in the unit ball $\mathcal{B}$ of $C^5(\mathcal{D};\mathbb{R})$ - we pose $\mathcal{M} = 1$. We now consider four given functions $F := (F_1, F_2, F_3, F_4) : \mathcal{D} \longrightarrow \mathbb{R}^4$ and, for any shift vector $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, we indicate their translations by

$$F_i^\lambda(x) := F_i(x) - \lambda_i \quad \forall i \in \{1,2,3,4\} \quad , \qquad F^\lambda := (F_1^\lambda, F_2^\lambda, F_3^\lambda, F_4^\lambda) \in \mathcal{D} \longrightarrow \mathbb{R}^4 \,,$$

and we set

$$\Phi := (F_1, F_2) : \mathcal{D} \longrightarrow \mathbb{R}^2 \quad , \qquad \Phi^\lambda := (F_1^\lambda, F_2^\lambda) : \mathcal{D} \longrightarrow \mathbb{R}^2$$

$$\Psi := (F_3, F_4) : \mathcal{D} \longrightarrow \mathbb{R}^2 \quad , \qquad \Psi^\lambda := (F_3^\lambda, F_4^\lambda) : \mathcal{D} \longrightarrow \mathbb{R}^2 \,.$$

For any given $\delta \in (0,1)$, we now fix a pair of parameters $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \backslash \mathcal{U}_\delta$ - where $\mathcal{U}_\delta$ is the bad set in (20.2.4) - together with a real number $\gamma > \dfrac{4}{3}$. We are interested in the sublevel

$$\mathtt{S}_\gamma := \{x \in \mathcal{D} : ||\Phi^\lambda(x)|| \leq \delta^\gamma\} \,,$$

where $||\cdot||$ is the standard euclidean norm in $\mathbb{R}^2$.

Since $\delta^\gamma < \delta$, Lemma 20.2.1 (for $n = m = 2$) ensures that if $x_0 \in \mathtt{S}_\gamma$ then

$$||D\Phi^\lambda(x_0)\xi|| > \delta^{2/3}||\xi|| \;\forall \xi \in \mathbb{R}^n \,. \tag{20.2.6}$$

[1] Actually, in [99] one is required $\mathcal{M} \geq 1$, but this condition is not really necessary once one slightly worsens the constant $\mathtt{C}$.

This implies that Φ^λ is locally invertible at x_0. One has $\Phi^\lambda(x_0) \subset \mathtt{B}_{\delta^\gamma}(0)$ by construction, $\mathtt{B}_{\delta^\gamma}(0)$ being the ball of radius δ^γ centered at the origin in the range of Φ^λ. We indicate by $(\Phi^\lambda_{x_0})^{-1}$ the inverse function, and we want to estimate the size of $(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))$. As it will be made clear in the sequel, this is a crucial step of the proof.

In order to do this, we make use of the quantitative local inversion Theorem I.0.1 and we estimate the parameters that appear in its statement. We start by taking a ball of radius $r > 0$ around x_0, which we denote by $\mathtt{B}_r(x_0)$, and, for any $x \in \mathtt{B}_r(x_0)$, we estimate the parameter ϱ as follows

$$
\begin{aligned}
\varrho &:= \sup_{x\in \mathtt{B}_r(x_0)} ||\mathbb{I} - (Df^\lambda(x_0))^{-1} D\Phi^\lambda(x)|| \\
&\leq \sup_{x\in \mathtt{B}_r(x_0)} \left\| \mathbb{I} - (D\Phi^\lambda(x_0))^{-1} \left(D\Phi^\lambda(x_0) + \int_0^1 \frac{d}{dt}[D\Phi^\lambda(tx + (1-t)x_0)]dt \right) \right\| \\
&\leq \left\| (D\Phi^\lambda(x_0))^{-1} \right\| \sup_{x\in \mathtt{B}_r(x_0)} \left\| \int_0^1 \nabla \left[D\Phi^\lambda(tx + (1-t)x_0)\right] \cdot (x - x_0) dt \right\| \\
&\lessdot \frac{\mathcal{M} r}{\min_{||\xi||=1} ||D\Phi^\lambda(x_0)\xi||} \lessdot \frac{r}{\delta^{2/3}}
\end{aligned}
\tag{20.2.7}
$$

where the last estimate is a consequence of (20.2.6) and of the fact that we have set $\mathcal{M} = 1$. The above calculation yields $\varrho \leq \frac{1}{2}$ if we fix

$$
r \doteq \delta^{2/3} .
$$

This choice also implies that the radius ζ of the ball $\mathtt{B}_\zeta(\Phi^\lambda(x_0)) \subset \Phi^\lambda(\mathtt{B}_r(x_0))$ in Theorem I.0.1 verifies the following estimate:

$$
\zeta := \frac{r(1-\varrho)}{||(D\Phi^\lambda)^{-1}(x_0)||} \doteq \delta^{2/3} \min_{||\xi||=1} ||D\Phi^\lambda(x_0)\,\xi|| \gtrdot \delta^{4/3} , \tag{20.2.8}
$$

where (20.2.6) was used once again in the last passage.

As $\gamma > 4/3$ by construction, by inequality (20.2.8) there exists a positive threshold $\delta_0 = \delta_0(\gamma, \mathcal{D}) < 1$ such that for any $0 < \delta < \delta_0$ one has $2\delta^\gamma < \zeta$ so that, for any $x_0 \in \mathtt{S}_\gamma$,

$$
\Phi^\lambda(x_0) \subset \mathtt{B}_{\delta^\gamma}(0) \subset \mathtt{B}_\zeta(\Phi^\lambda(x_0)) ,
$$

and $\mathtt{B}_{\delta^\gamma}(0)$ is contained in the domain of the inverse function.

We are now able to estimate the size of $(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))$. Indeed, the Lipschitz constant of the inverse function is given by the parameter L_φ in Theorem I.0.1, that in our case is bounded by

$$
L_{(\Phi^\lambda_{x_0})^{-1}} \leq \frac{||(D\Phi^\lambda(x_0))^{-1}||}{1-\varrho} \leq \frac{2}{\min_{||\xi||=1} ||D\Phi^\lambda(x_0)\xi||} \leq \frac{2}{\delta^{2/3}} ,
$$

so that

$$\text{meas}\left[(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))\right] \lessdot (L_{(\Phi^\lambda_{x_0})^{-1}}\,\delta^\gamma)^2 \doteq \delta^{2\gamma-4/3}\ .$$

This estimate is crucial because it allows to infer a bound on the size of $\Psi(\mathtt{S}_\gamma)$. Infact, the image of the set $(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))$ through the function Ψ satisfies

$$\text{meas}\left\{\Psi\left[(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))\right]\right\} \lessdot \delta^{2\gamma-4/3}\ , \tag{20.2.9}$$

due to the fact that the Lipschitz constant is bounded by the first derivatives of Ψ and that Ψ belongs to the unit ball $\mathcal{B}$. Then, we must estimate the number of inverse images of the kind $(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))$, which depends on the number $N(\delta)$ of balls in $\mathtt{S}_\gamma$ on which Φ^λ can be inverted. $N(\delta)$ admits the upper bound

$$N(\delta) \le \frac{\text{meas}\ \mathcal{D}}{\text{meas}\ \mathtt{B}_r(x_0)} \lessdot \frac{1}{r^2} \doteq \frac{1}{\delta^{4/3}}\ . \tag{20.2.10}$$

The size of $\Psi(\mathtt{S}_\gamma)$ can therefore be bounded from above by taking (20.2.9) and (20.2.10) into account, namely

$$\text{meas}\ \Psi(\mathtt{S}_\gamma) \le N(\delta)\times\text{meas}\left[\Psi((\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0)))\right] \lessdot \delta^{2(\gamma-4/3)}\ . \tag{20.2.11}$$

Theorem 20.2.1 is a consequence of estimate (20.2.11). To see this, we impose to the shift parameters (λ_3,λ_4) to lie outside of a neighborhood of size δ^γ around $\Psi(\mathtt{S}_\gamma)$, which we denote by

$$\mathcal{W}_{\delta,\gamma} := \{y\in\mathbb{R}^2\ :\ ||y-\Psi(\mathtt{S}_\gamma)|| \le \delta^\gamma\}\ .$$

By (20.2.11), the size of set $\mathcal{W}_{\delta,\gamma}$ can be obtained by enlarging the image sets $\Psi((\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0)))$ of a width δ^γ, and by multiplying by $N(\delta)$. By formula (20.2.9) we have that

$$\text{diam}\ \Psi[(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))] \lessdot \delta^{\gamma-2/3}$$

so that, since $\delta<1$ and consequently $\delta^\gamma<\delta^{\gamma-2/3}$, we have

$$\text{meas}\ \{y\in\mathbb{R}^2\ :\ \text{dist}\ (y,\Psi[(\Phi^\lambda_{x_0})^{-1}(\mathtt{B}_{\delta^\gamma}(0))]) \le \delta^\gamma\} \lessdot \delta^{2(\gamma-2/3)}$$

and

$$\text{meas}\ \mathcal{W}_{\delta,\gamma} \lessdot N(\delta)\times\delta^{2(\gamma-2/3)} \lessdot \delta^{2(\gamma-4/3)}\ . \tag{20.2.12}$$

Hence, by taking $(\lambda_3,\lambda_4)\in\mathbb{R}^2\backslash\mathcal{W}_{\delta,\gamma}$, by the above construction and by the equivalence of norms we have that for all $x_0\in\mathtt{S}_\gamma$

$$||\Psi^\lambda(x_0)|| \gtrdot |F^\lambda_3(x_0)|+|F^\lambda_4(x_0)| \gtrdot ||(\lambda_3,\lambda_4)-\Psi(\mathtt{S}_\gamma)|| \gtrdot \delta^\gamma\ . \tag{20.2.13}$$

This implies that, taken $(\lambda_1,\lambda_2)\times(\lambda_3,\lambda_4)\in(\mathbb{R}^2\backslash\mathcal{U}_\delta)\times(\mathbb{R}^2\backslash\mathcal{W}_{\delta,\gamma})$, as $\mathcal{D}$ is compact, estimate

$$\min_{x\in\mathcal{D}}\left(|F^\lambda_1(x)|+|F^\lambda_2(x)|+|F^\lambda_3(x)|+|F^\lambda_4(x)|\right) \ge \mathtt{C}_2\,\delta^\gamma \tag{20.2.14}$$

holds true for some absolute constant $\mathtt{C}_2$ depending only on the form of $\mathcal{D}$. Infact, we indicate by $x^\star \in \mathcal{D}$ the point at which the minimum at the l.h.s. of formula (20.2.14) is attained. Then, either $x^\star \notin \mathtt{S}_\gamma$, in which case

$$|F_1^\lambda(x^\star)| + |F_2^\lambda(x^\star)| \geq \delta^\gamma$$

by construction, or $x^\star \in \mathtt{S}_\gamma$ and

$$|F_3^\lambda(x^\star)| + |F_4^\lambda(x^\star)| \gtrdot \delta^\gamma$$

by inequality (20.2.13). $\mathtt{C}_2$ is therefore the minimum between 1 and the implicit constant in (20.2.13).

We also notice that the choice $\gamma = \dfrac{19}{12}$, together with (20.2.12), entails

$$\text{meas}\ \mathcal{W}_{\delta,19/12} \doteqdot \sqrt{\delta}\ . \tag{20.2.15}$$

The thesis follows by choosing $\eta := \sqrt{\delta}$, $\mathcal{V}_\eta := \mathcal{U}_\delta$, $\mathcal{E}_\eta := \mathcal{W}_{\delta,19/12}$, and $\mathtt{C}_1$ as the maximum between the constant $\mathtt{C}$ in Lemma 20.2.1 and the implicit constant in (20.2.15).

Remark 20.2.4. The above reasonings can be extended to the case of functions depending on an arbitrary number n of variables. As it has already been pointed out in subsection (20.1), in that case, one considers the functions F_i, with $i \in \{1, \dots, 2(\lfloor n/2 \rfloor + 1)\}$ - together with the translations $F_i^\lambda := F_i - \lambda_i$, of shift vector $\lambda \in \mathbb{R}^{2(\lfloor n/2 \rfloor+1)}$ - and applies the arguments of this section to

$$\begin{aligned}
\Phi &:= (F_1, F_2, \dots, F_{\lfloor n/2 \rfloor+1}) : \mathcal{D} \longrightarrow \mathbb{R}^{\lfloor n/2 \rfloor+1}\\
\Phi^\lambda &:= (F_1^\lambda, F_2^\lambda, \dots, F_{\lfloor n/2 \rfloor+1}^\lambda) : \mathcal{D} \longrightarrow \mathbb{R}^{\lfloor n/2 \rfloor+1}\\
\Psi &:= (F_{\lfloor n/2 \rfloor+2}, F_{\lfloor n/2 \rfloor+3}, \dots, F_{2(\lfloor n/2 \rfloor+1)}) : \mathcal{D} \longrightarrow \mathbb{R}^{\lfloor n/2 \rfloor+1}\\
\Psi^\lambda &:= (F_{\lfloor n/2 \rfloor+2}^\lambda, F_{\lfloor n/2 \rfloor+3}^\lambda, \dots, F_{2(\lfloor n/2 \rfloor+1)}^\lambda) : \mathcal{D} \longrightarrow \mathbb{R}^{\lfloor n/2 \rfloor+1}\ .
\end{aligned}$$

Chapter 21

Further heuristics

In this paragraph, we will consider a mechanical system

$$H(I,\theta) = \frac{I^2}{2} + \varepsilon f(I,\theta) \quad , \qquad f \in C^{\omega}(D \times \mathbb{T}^n) \, . \tag{21.0.1}$$

The extension of the heuristic arguments of this paragraph to the general analytic nearly-integrable Hamiltonian $H(I,\theta) = h(I) + \varepsilon f(I,\theta)$ will be discussed in the following section, where a rigorous result is presented.

21.1 Application of KAM Theory

As it was the case in the study of perturbations depending only on the angles (see section 19.3), also in this context the strategy depends on the size of the vector $k \in \mathbb{Z}^n \backslash \{0\}$ generating the resonance. In particular, we will distinguish between a finite number of "low modes" associated to vectors $k \in \mathbb{Z}^n \backslash \{0\}$ satisfying

$$k_1 > 0 \quad , \qquad \gcd(k_1, \dots, k_n) = 1 \quad , \qquad 1 \le |k| \lesssim 1/s$$

and an infinite number of "high modes" verifying

$$k_1 > 0 \quad , \qquad \gcd(k_1, \dots, k_n) = 1 \quad , \qquad 1/s << |k| \le K \, .$$

Low modes The low modes are treated in a similar way to the case of a perturbation $f \in C^{\omega}(\mathbb{T}^n)$ depending only on the angles. In particular, one has to deal with a finite number of phase portraits which can be in principle very complicated. Then, technical non-perturbative results are needed in order to check Kolmogorov's non-degeneracy condition for these systems. This latter aspect is highly non-trivial and is essentially non-perturbative. It will not be discussed here (see [22]).

High modes. By the arguments developed in the previous sections, for any vector $k \in \mathbb{Z}^n \backslash \{0\}$ associated to a "high mode resonance" satisfying (20.1.13), the first goal is to build action-angle coordinates for the integrable approximation (see (20.1.3))

$$\frac{I^2}{2} + 2\varepsilon \sum_{j\in\mathbb{Z},|j|\le\ell} |f_{jk}(I)| \cos(j(k\cdot\theta) + \theta^{jk}) \tag{21.1.1}$$

inside the simple resonant domain D_1^k.

Then, one has to check if Kolmogorov's non-degeneracy condition is satisfied by the integrable Hamiltonian in the new action variables, in order to establish the existence of secondary tori in D_1^k.

As it was the case in the original problem, where the perturbation f depended only on the angles, the fact that near high mode resonances the system is always close to the same Hamiltonian (21.1.1) up to a suitably small remainder simplifies the problem dramatically, as it eliminates the difficulty of having a uniform control on the parameters of an infinite number of different systems.

When $f \in C^\omega(D\times\mathbb{T}^n)$, two main differences occur w.r.t. the original case $f \in C^\omega(\mathbb{T}^n)$ considered in [22].

1. In general, for $\ell \ge 2$, bifurcations appear in the topological properties of the phase space of the integrable approximation

$$2\varepsilon \sum_{j\in\mathbb{Z}\setminus\{0\},|j|\le\ell} |f_{jk}(I)| \cos(j(k\cdot\theta) + \theta^{jk}) \ .$$

To fix the ideas, we choose $\ell = 2$, we consider the example of Hamiltonian

$$\frac{y^2}{2} + a\cos(x) + b\cos(2x) \quad , \qquad a, b > 0\ , \ \ (y, x) \in \mathbb{R}^2$$

and we plot the associated phase space for different values of a and b.

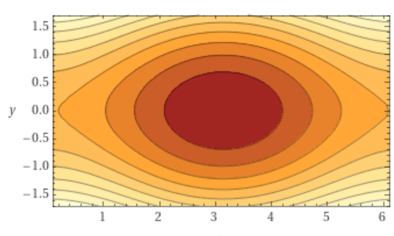

Figure 21.1: $a = 0.5, b = 0.01$

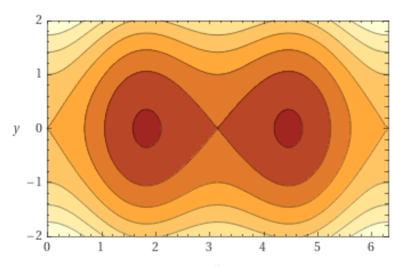

Figure 21.2: $a = 0.5, b = 0.5$

In the first figure, which is topologically equivalent to that of the simple pendulum, we see that there exists only class of homotopically trivial curves associated to librations. In the second figure, where the two addends $a\cos(x)$ and $b\cos(2x)$

have the same magnitude, instead, two elliptic points and more separatrices than in the previous case appear, so that different kinds of librations are possible, depending on the value of the energy.

2. Degenerated critical points occur. For example, system

$$\frac{y^2}{2} + a(\cos(kx) + b\cos(2kx)) \quad a, b \geq 0 \tag{21.1.2}$$

has a degenerated critical point at the origin for $a = 0$ of for $b = \frac{1}{4}$.

On the contrary, it can be shown that in the original setting, when $f \in C^{\omega}(\mathbb{T}^n)$ and the system is conjugated to a "simple pendulum", degenerated critical points do not occur.

Without entering into details (for which we refer to [22]), we observe that the presence of degenerated critical ponts is an obstacle to the verification of the non-degeneracy conditions needed in order to apply KAM theory. However, as for a given system there is only a finite number of these singularities, one may "avoid" them in a quantitative way by making use again of Yomdin and Comte's quantitative Morse-Sard's Theory. Heuristically speaking, the idea would be to prove that - for most potentials - when the first derivatives are close to zero, then the second derivatives are not.

21.2 Extension to a generic integrable part

In this section, we briefly discuss a result which constitutes an important step in order to extend our reasonings from mechanical Hamiltonians of the form

$$H(y, x) = \frac{y^2}{2} + \varepsilon f(y, x) \quad , \qquad f \in C^{\omega}(D \times \mathbb{T}^n)\,. \tag{21.2.1}$$

to more general Hamiltonians of the kind

$$H(y, x) = h(y) + \varepsilon f(y, x) \quad , \qquad H \in C^{\omega}(D \times \mathbb{T}^n)\,. \tag{21.2.2}$$

We start by stating the following

Lemma 21.2.1 (see ref. [24])**.** *For any $k \in \mathbb{Z}^n$ verifying (20.1.13), there exists a matrix $\mathsf{A} \in \mathbb{Z}^{(n-1)\times n}$ such that*[1]

$$\mathsf{B} := \begin{pmatrix} k \\ \mathsf{A} \end{pmatrix} = \begin{pmatrix} k_1 \cdots k_n \\ \mathsf{A} \end{pmatrix} \in \mathrm{SL}(n, \mathbb{Z})\,,$$

$$|\mathsf{A}|_{\infty} \leq |k|_{\infty}\,, \;\; |\mathsf{B}|_{\infty} = |k|_{\infty}\,, \;\; |\mathsf{B}^{-1}|_{\infty} \leq (n-1)^{\frac{n-1}{2}}\, |k|_{\infty}^{n-1}\,.$$

[1] $|M|_{\infty}$, with M matrix (or vector), denotes the maximum norm $\max_{ij} |M_{ij}|$ (or $\max_i |M_i|$).

Then, we observe that - close to a simple resonance generated by a vector $k \in \mathbb{Z}^n$ verifying (20.1.13), namely close to

$$\mathcal{R}_k := \{y \in D | k \cdot \partial_y h(y) = 0\} \cap D_1 ,$$

where the block of simple[2] resonances D_1 was introduced in (19.2.2), one can introduce the adapted symplectic change of variables

$$\varphi(x, y) : D \times \mathbb{T}^n \longrightarrow \mathbb{R}^n \times \mathbb{T}^n \quad , \qquad (y, x) \longmapsto \begin{cases} I = (\mathsf{B}^\dagger)^{-1} y \\ \theta = \mathsf{B} x \, . \end{cases} \tag{21.2.3}$$

We indicate by $\mathsf{H}(I, \theta) := H \circ \varphi^{-1}(I, \theta) = \mathsf{h}(I) + \varepsilon \mathsf{f}(I, \theta)$ Hamiltonian (21.2.2) written in the new variables. It is plain to check that if $y_0 \in \mathcal{R}_k$, then in the new coordinates (21.2.3) one has $\partial_{I_1} \mathsf{h}(I)|_{I_0} = 0$, where I_0 is the image of y_0 through the inverse of transformation (21.2.3). Namely, the image of the simple resonant set $\mathcal{R}_k$ through (21.2.3) is constituted by critical points for the functions $I_1 \longmapsto \mathsf{h}(I_1, I_2, \dots, I_n)$, which are parametrized by the dummy actions $I_2, \dots, I_n$.

Then, we introduce the following definitions

Definition 21.2.1. A $C^2(\mathbb{T}, \mathbb{R})$ Morse function F with distinct critical values is called **β–Morse**, with $\beta > 0$, if

$$\min_{\theta \in \mathbb{T}} \big(|F'(\theta)| + |F''(\theta)|\big) \geq \beta \, , \quad \min_{i \neq j} |F(\theta_i) - F(\theta_j)| \geq \beta \, , \tag{21.2.4}$$

where $\theta_i \in \mathbb{T}$ are the critical points of F.

Definition 21.2.2. Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^{n-1}$ be a bounded domain, $\mathtt{R} > 0$ and $\mathcal{D} := (-\mathtt{R}, \mathtt{R}) \times \hat{\mathcal{D}}$. We say that the real-analytic Hamiltonian $\mathcal{H}$ is in standard form with respect to standard symplectic variables $(I_1, \theta_1) \in (-\mathtt{R}, \mathtt{R}) \times \mathbb{T}$ and 'external actions'

$$\hat{I} := (I_2, ..., I_n) \in \hat{\mathcal{D}}$$

if $\mathcal{H}$ has the form

$$\mathcal{H}(I, \theta_1) = \big(1 + \nu(I, \theta_1)\big) I_1^2 + \mathsf{G}(\hat{I}, \theta_1) \, , \tag{21.2.5}$$

where:

- ν and G are real-analytic functions defined on, respectively, $\mathcal{D}_r \times \mathbb{T}_s$ and $\hat{\mathcal{D}}_r \times \mathbb{T}_s$ for some $0 < r \leq \mathtt{R}$ and $s > 0$;

- G has zero average and there exists a function G_0 (the 'reference potential') depending only on θ_1 such that, for some[3] $\beta > 0$,

$$\mathsf{G}_0 \text{ is } \beta\text{–Morse} \, , \qquad \langle \mathsf{G}_0 \rangle = 0 \, ; \tag{21.2.6}$$

[2] We stress that, in the above expression, if the frequency vector $\partial_y h(y)$ verifies $k' \cdot \partial_y h(y) =$ for some other $k' \in \mathbb{Z}^n$ satisfying (20.1.13), then k' must be parallel to k.

[3] Recall Definition 21.2.1.

– the following estimates hold:

$$\begin{cases} \sup_{\mathbb{T}_s} |\mathsf{G}_0| \le \varepsilon\,, \\ \sup_{\hat{D}_r\times\mathbb{T}_s} |\mathsf{G}-\mathsf{G}_0| \le \varepsilon\mu\,, \quad \text{for some} \quad 0<\varepsilon\le r^2/2^{16}\,,\ \ 0\le\mu<1\,, \\ \sup_{D_r\times\mathbb{T}_s} |\nu| \le \mu\,. \end{cases} \tag{21.2.7}$$

Remark 21.2.1. The Hamiltonian in standard form (21.2.5) retains the basic features of the standard pendulum, more precisely of a natural system with a generic periodic potential, having, in particular all equilibria on the $I_1 = 0$ axis in the (I_1, θ_1)–phase space.

It is a remarkable fact that, close to the simple resonance $\mathcal{R}_k$ introduced above, the Hamiltonian $\mathsf{H}(I,\theta) = \mathsf{h}(I) + \varepsilon\mathsf{f}(I,\theta)$ written in the resonant variables (21.2.3) associated to $\mathcal{R}_k$ can be symplectically conjugated to a Hamiltonian in standard form, provided that the perturbation f is Morse w.r.t. the angle. Namely, in ref. [25] it is proved that

Proposition 21.2.1. *Let $\mathsf{H}(I,\theta) = \mathsf{h}(I) + \varepsilon\mathsf{f}(I,\theta)$ be a real-analytic function and assume that at $I_0 \in D$ the function $I_1 \longmapsto \mathsf{h}(I_1, \hat{I})$ has a non–degenerated critical point*[4]*. Assume also that $\theta_1 \longmapsto \mathsf{f}(I_0, \theta_1)$ is a Morse function with distinct critical values. Then, for ε small enough, H is symplectically conjugated to a Hamiltonian in standard form in a (ε–independent) neighborhood of $I_0 \times \mathbb{T}^n$.*

[4] Explicitly: $\partial_{I_1}\mathsf{h}(I_0) = 0$ and $\partial^2_{I_1^2}\mathsf{h}(I_0) \neq 0$.

Part V

Annex - On the algebraic properties of exponentially stable integrable Hamiltonian systems

Abstract

Steepness is a geometric property which, together with complex-analyticity, is needed in order to insure stability of a near-integrable hamiltonian system over exponentially long times. Following a strategy developed by Nekhoroshev, we construct sufficient algebraic conditions for steepness for a given function that involve algebraic equations on its derivatives up to order five. The underlying analysis suggests some interesting considerations on the genericity of steepness and represents a first step towards the construction of sufficient conditions for steepness involving the derivatives of the studied function up to an arbitrary order.

Notice

This part of the thesis is the study of genericity and criteria for steepness for polynomials with depending on a small number of variables and of degree less or equal than five. It was the original starting point for the results of Part I, which contains a way deeper and general discussion. However, this part is somehow still interesting as it contains some examples of integrable Hamiltonians on which the explicit conditions for steepness are tested. Moreover, it can be a good way to get introduced to the complicated calculations of Part I.

Moreover, this part has already been published. Its reference is:

S. Barbieri, *On the algebraic properties of exponentially stable integrable Hamiltonian systems*, Annales de la Faculté des Sciences de Toulouse, 31 (6), pp. 1365-1390, 2022.

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Chapter 22

Introduction

Hamiltonian formalism is the natural setting appearing in the study of many physical systems. In the simplest case, we consider the motion of a point on a Riemannian manifold $\mathcal{M}$, called configuration manifold, governed by Newton's second law ($\ddot{q} = -\nabla U(q)$ for a potential function U in the euclidean case, with q a system of local coordinates for $\mathcal{M}$). This system can be transformed by duality thanks to Legendre's transformation and reads

$$\dot{p} = -\partial_q H(p,q) \quad , \qquad \dot{q} = \partial_p H(p,q) \quad ,$$

where $H(p,q)$ is a real differentiable function on the cotangent bundle $T^*\mathcal{M}$, classically called Hamiltonian, and p is the coordinate conjugated to q. Systems integrable by quadrature are an important class of Hamiltonian systems. By the classical Liouville-Arnol'd Theorem, under general topological assumptions, an integrable system depending on $2n$ variables (n degrees of freedom) can be conjugated to a Hamiltonian system on the cotangent bundle of the n-dimensional torus $\mathbb{T}^n$, whose equations of motion take the form

$$\dot{I} = -\partial_\vartheta h(I) = 0 \quad , \qquad \dot{\vartheta} = \partial_I h(I) \quad ,$$

where $(I,\vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ are called action-angle coordinates. Therefore, the phase space for an integrable system is foliated by invariant tori carrying the linear motions of the angular variables (called quasi-periodic motions). Integrable systems are exceptional, but many important physical problems are governed by Hamiltonian systems which are close to integrable. Namely, the dynamics of a near-integrable Hamiltonian system is described by a Hamiltonian function whose form in action-angle coordinates $(I,\vartheta) \in \mathbb{R}^n \times \mathbb{T}^n$ reads

$$H(I,\vartheta) := h(I) + \varepsilon f(I,\vartheta) \, ,$$

where ε is a small parameter. The structure of the phase space for this kind of systems can be inferred with the help of Kolmogorov-Arnol'd-Moser (KAM) theory. Namely, under a general non-degeneracy condition for h, a Cantor set of positive measure of invariant tori carrying quasi-periodic motions for the integrable flow persists under a

suitably small perturbation (see e.g. ref. [5], [46]).
For systems with three or more degrees of freedom, KAM theory yields little information about trajectories lying in the complementary of such Cantor set, where instabilities can occur (see e.g. ref. [3]). However, in a series of articles published during the seventies (see ref. [95], [96]) Nekhoroshev proved an effective result of stability for an open set of initial conditions holding over a time which is exponentially long in the inverse of the size ε of the perturbation, provided that the Hamiltonian is analytic and that its integrable part satisfies a generic transversality property known as *steepness*.
From a more technical point of view, steepness is defined as follows:

Definition 1. Let $\mathcal{A}$ be an open set of $\mathbb{R}^n$ and $h : \mathcal{A} \longrightarrow \mathbb{R}$ a smooth function. h is *steep* at $I := (I_1, ..., I_n) \in \mathcal{A}$ if $\nabla h(I) \neq 0$ and if, for any $m = 1, ..., n-1$, there exist constants $C_m > 0$, $\delta_m > 0$ and $\alpha_m > 1$ such that, for all m-dimensional affine subspace Λ_m^I orthogonal to $\nabla h(I)$, the gradient of the restriction of h to Λ_m^I, which we denote by $\nabla(h|_{\Lambda_m^I})$, satisfies

$$\max_{0\le\eta\le\xi}\left(\min_{I'\in\Lambda_m^I,\ ||I-I'||=\eta} ||\nabla(h|_{\Lambda_m^I})(I')||\right) > C_m\xi^{\alpha_m}\ , \quad \forall\xi\in(0,\delta_m]\ . \tag{22.0.1}$$

The constants C_m and δ_m are called the steepness coefficients of h, whereas the α_m are its steepness indices. In particular, in the analytic case, a function is steep if and only if, on any affine hyperplane Λ_m^I, there exists no curve γ with one endpoint in I such that the restriction $\nabla(h|_{\Lambda_m^I})$ identically vanishes on γ, as is showed in ref. [98]. From a heuristic point of view, for any value $m \in \{1, ..., n-1\}$ the gradient ∇h must "bend" towards Λ_m^I when "travelling" along the curve $\gamma \in \Lambda_m^I$, so that critical points for the restriction of h to Λ_m^I must not accumulate (see ref. [98]). Finally, h is said to be steep in a given domain if it is steep at each point of such set with uniform indices and coefficients.
With such notion, Nekhoroshev's effective result of stability reads

Theorem (Nekhoroshev, 1977) 1. Consider a near-integrable system with Hamiltonian $H(I,\vartheta) := h(I)+\varepsilon f(I,\vartheta)$ analytic in some complex neighborhood of $B_r\times\mathbb{T}^n$, where B_r is the open ball of radius r in $\mathbb{R}^n$, and suppose h steep. Then there exist positive constants $a, b, \varepsilon_0, C_1, C_2$ such that, for any $\varepsilon \in [0, \varepsilon_0)$ and for any initial condition not too close from the boundary, one has $|I(t) - I(0)| \le C_2\varepsilon^a$ for any time t satisfying $|t| \le C_1 \exp\left(\varepsilon^{-b}\right)$.

Such result also holds under the weaker regularity assumption that the Hamiltonian is in the Gevrey class (see ref. [87]) and by requiring steepness to be verified only on those subspaces which are spanned by integer vectors satisfying suitable arithmetic conditions (see refs. [71], [99]). However, one cannot get completely rid of the steepness hypothesis since examples of instability over times of order $1/\varepsilon$ may be constructed in case such property is not satisfied on a subspace spanned by integer vectors (see ref. [98], [34]). Therefore, a crucial step in order to establish stability over exponentially long times for a near-integrable Hamiltonian system consists in building a suitable steep integrable approximation. This aspect is important when trying to apply Nekhoroshev's estimates to concrete examples, as it is shown for example in refs. [32] and [102]. As we shall see, the steepness property is generic, both in measure and topological sense. However, since its definition is not constructive, it is difficult to directly establish wether a given function is steep or not. Fortunately, Nekhoroshev provided in [96] a scheme which, in principle, allows to deduce explicit sufficient algebraic conditions for steepness involving the derivatives of the studied function up to an arbitrary order. In particular, let us define the r-jet $P_I(h, r, n)$ of a smooth function h of n variables at I as the vector containing all the coefficients of the Taylor polynomial of h at I up to order r, with the exception of the constant term, namely

$$P_I(h, r, n) := \left\{ \frac{1}{\mu!} \frac{\partial^\mu h}{\partial I^\mu},\ 1 \le |\mu| \le r \right\} ,$$

where $\mu := (\mu_1, ...\mu_n)$ is a multi-index of naturals and $|\mu| = \sum_{i=1}^n \mu_i$.
With this definition, one can pass to the quotient in the set of smooth functions and consider a representative of the class of smooth functions of n variables having the same r-jet at I. We also denote by $\mathcal{P}_I(r, n)$ the polynomial space of the r-jets of smooth functions of n variables calculated at I. Nekhoroshev showed that, for any $r \ge 2$, one can construct a semi-algebraic set whose closure contains the r-jets of all non-steep functions with non-zero gradient at I. Namely, we have the following

Theorem (Nekhoroshev, 1979) 2. For any $n \ge 2$ and $r \ge 2$, there exists a semi-algebraic set $\sigma_n^r(I) \subset \mathcal{P}_I(r, n)$, whose closure is denoted by $\Sigma_n^r(I)$, such that any given function h satisfying:

1. $h \in C^{2r-1}$ in a neighborhood of I ,
2. $\nabla h(I) \neq 0$,
3. $P_I(h, r, n) \in \mathcal{P}_I(r, n) \backslash \Sigma_n^r(I)$,

is steep in some neighborhood of I.
Moreover, for any $m = 1, ..., n-1$, one has

$$\operatorname{codim} \Sigma_n^r \ge \begin{cases} \max\left\{0, r - 1 - \dfrac{n(n-2)}{4}\right\}, & \text{if } n \text{ is even} \\ \max\left\{0, r - 1 - \dfrac{(n-1)^2}{4}\right\}, & \text{if } n \text{ is odd} \end{cases} \tag{22.0.2}$$

and the steepness indices α_m of h are superiorly bounded by

$$\bar{\alpha}_m := \begin{cases} \max\left\{1, 2r - 3 - \dfrac{n(n-2)}{2} + 2m(n-m-1)\right\}, & \text{if } n \text{ is even} \\ \max\left\{1, 2r - 3 - \dfrac{(n-1)^2}{2} + 2m(n-m-1)\right\}, & \text{if } n \text{ is odd} \end{cases} . \tag{22.0.3}$$

As Nekhoroshev points out in his discussion, such result implies a stratification in the space of jets: the strata $\Sigma_n^r(I)$, with $r \geq 2$, are semialgebraic sets of increasing codimension. Hence, as expression (22.0.2) shows, for fixed n and sufficiently high r, the steepness property is generic in $\mathcal{P}_I(r,n)$. Moreover, for fixed values of r and n, in addition to non-steep functions, also all steep functions with steepness indices greater than $\bar{\alpha}_m$ are contained in the stratus $\Sigma_n^r(I)$. In other words, for increasing values of r, the complementary of $\Sigma_n^r(I)$ contains more and more jets of steep functions and the steepness indices of such functions are superiorly bounded by a quantity $\bar{\alpha}_m$ which increases linearly with r. A way to obtain sufficient conditions for steepness in the space of jets at a fixed order r consists therefore in knowing the explicit form of the stratus $\Sigma_n^r(I)$ or the form of some closed set containing it: a function whose r-jet lies outside such set is steep. The sets $\Sigma_n^2(I)$, for any $n \geq 2$, have been explicitly described by Nekhoroshev in references [95] and [96]. Before stating Nekhoroshev's results, we denote by

$$h_I^k[v^1, ..., v^k] = \sum_{i_1,...,i_k=1}^{n} \frac{\partial^k h}{\partial I_{i_1} ... \partial I_{i_k}}(I) v^1_{i_1} ... v^k_{i_k}$$

the k-th order multilinear form corresponding to the k-th coefficient of the Taylor expansion of a function h which is k-times continuously differentiable in a neighborhood of I. We also give the following

Definition 2. For $r \in \mathbb{N}$, $r \geq 2$, a function h of class C^r in a neighborhood of a point I is said to be r-jet non-degenerate if the system

$$h^1[v] = 0\ ;\ \ h^2[v,v] = 0\ ,\ \ ...\ ,\ \ h^r[v, ..., v] = 0$$

admits only the trivial solution $v = 0$. If this is not the case, h is said to be r-jet degenerate.

With such setup, Nekhoroshev proved that, in the space of jets of order two, one has $P_I(h,2,n) \in \mathcal{P}_I(2,n) \backslash \Sigma_n^2(I)$ if and only if h is two-jet degenerate. Such condition is equivalent to requiring that h is quasi-convex (i.e. convex on level sets) at I and Theorem 2 implies that all quasi-convex functions in C^3 class around a non-critical point I are steep in a neighborhood of such point. In a similar way, Nekhoroshev found a sufficient condition for steepness involving the derivatives of order three: namely, if a function $h \in C^5$ around a non-critical point I is three-jet non-degenerate at I, then h is steep in a neighborhood of such point. As we shall see in subsection 27.3, such result is more general than the conditions that can be inferred on the jets of order three by simply following the scheme of Theorem 2, since it applies to a wider set of functions. Its proof is not found in Nekhoroshev's works (see refs. [94] and [96]) and it has been explicitly written in an analytic way in [47] for systems with any number of degrees of freedom. As we shall show in subsection 27.3, the fact that the three-jet non degeneracy of a given function depending on $n = 2, 3, 4, 5$ coordinates implies its steepness comes out as a straightforward corollary of the algebraic structure of the equations that define the sets $\Sigma^r_n(I)$, for $r = 4, 5$ and $n = 2, 3, 4, 5$. Following such discussion, we conjecture that the algebraic properties of the sets $\Sigma^r_n(I)$, for any values of r and n, can be used to prove the steepness of all three-jet non-degenerate functions depending on an arbitrary number of variables; this would constitute an alternative proof to the one used in [47]. However, the algebraic form of the strata $\Sigma^r_n(I)$, for $r \geq 4$, cannot be expressed so straightforwardly as in the cases $r = 2, 3$. In [111] the authors were able to build closed sets containing the strata $\Sigma^4_n(I)$ for $n = 2, 3, 4$ by exploiting Nekhoroshev's strategy. For $n \geq 5$ and $r = 4$, on the other hand, Nekhoroshev's scheme turns out not to be helpful since it yields conditions for steepness which are stronger than three-jet non degeneracy.

In this work, we develop the scheme in [111] and we build closed sets containing $\Sigma^5_n(I)$ for $n = 2, 3, 4, 5$. This allows us to formulate new explicit conditions for steepness involving the five-jet of a given function. Similarly to the case considered in [111], the constraints we find are useful only in the case of systems with $n = 2, 3, 4, 5$ degrees of freedom, as we shall discuss in subsection 27.1. Moreover, we slightly modify the construction in [111] so to get rid of some hypotheses of non-degeneracy on the hessian matrix of the function whose steepness is being tested. Furthermore, this work can be seen as a first step towards the formulation of sufficient conditions for steepness in the space of jets of arbitrary order. Indeed, a comparison on the equations defining the 'bad' sets $\sigma^r_n(I)$ defined in Theorem 2, for $r = 4, 5$, suggests hints on the algebraic structure of $\sigma^r_n(I)$ for any value of r, which shall be studied in detail in a further work. By formula (22.0.2), this would allow to obtain generic conditions for steepness for functions depending on an arbitrary number of degrees of freedom. Actually, if the explicit expression of the sets $\sigma^n_r(I)$ were known for all $r, n \in \mathbb{N}$, for any fixed value of n one should then simply find the minimal order r^*, depending on n, for which the codimension of the bad set $\sigma^r_n(I)$ is positive. At that point, steepness would be generic in the space of jets of order r^* and a way to test steepness of a given function would be to see if its r^*-jet belongs to the complementary of the closure of $\sigma^{r^*}_n(I)$.

This part is organized as follows: in section 23 we state our results, whereas in section 24 we test such conditions on a couple of polynomial examples. Section 25 is dedicated to an overview on Nekhoroshev abstract strategy to construct sets $\sigma^r_n(I)$, section 26 contains the proofs of the statements in section 23 and, finally, section 27 contains some remarks and a short discussion on the possible developements of this work.

Chapter 23

Results

Below, we state our results separately for each of the possible values of the number of degrees of freedom n.
As a matter of notation, for fixed n and for any collection of $m \in \{1, ..., n-1\}$ vectors $v_1, ..., v_m$ in $\mathbb{R}^n$, we shall indicate by $rk(v_1, ..., v_m)$ the linear rank of the matrix $(v_1, ..., v_m)$ generated by such collection.

For $n = 2$ we have

Theorem 3. Let $\mathcal{A}$ an open set of $\mathbb{R}^2$ and $h : \mathcal{A} \longrightarrow \mathbb{R}$ a smooth function. Let $I \in \mathcal{A}$ a point such that $\nabla h(I) \neq 0$. If h is five-jet non-degenerate at I, then h is steep in some neighborhood of I.

For $n = 3$ we have

Theorem 4. Let $\mathcal{A}$ an open set of $\mathbb{R}^3$ and $h : \mathcal{A} \longrightarrow \mathbb{R}$ a smooth function. Let $I \in \mathcal{A}$ a point such that $\nabla h(I) \neq 0$. If

1. h is five-jet non-degenerate at I;
2. for any $v \neq 0$ such that h is three-jet degenerate at I, any vector u solving system

$$h^1_I[u] = 0\ ;\ \ h^2_I[u, v] = 0\ ;\ \ h^2_I[u, u]h^4_I[v, v, v, v] = 3(h^3_I[v, v, u])^2$$

   satisfies $rk(u, v) < 2$;

then h is steep in some neighborhood of I.

For $n = 4$ we have

Theorem 5. Let $\mathcal{A}$ an open set of $\mathbb{R}^4$ and $h : \mathcal{A} \longrightarrow \mathbb{R}$ a smooth function. Let $I \in \mathcal{A}$ a point such that $\nabla h(I) \neq 0$.
If

1. h is five-jet non-degenerate at I;
2. for all $v \neq 0$ such that h is three-jet degenerate at I, any vector u solving system

$$\begin{cases} h_I^1[u]=0\ ;\ \ h_I^2[u,v]=0\ ;\ \ h_I^2[u,u]h_I^4[v,v,v,v]=3(h_I^3[v,v,u])^2 \\ 15(h_I^3[v,v,u])^2 h_I^3[u,u,v]+h_I^5[v,v,v,v,v](h_I^2[u,u])^2 \\ \ =10h_I^4[v,v,v,u]h_I^3[u,v,v]h_I^2[u,u] \end{cases} \tag{23.0.1}$$

   satisfies $rk(u,v)<2$;

3. for all $v \neq 0$ such that h is three-jet degenerate at I, any couple of vectors (u,w) solving

$$h_I^1[u]=0\ ;\ \ h_I^1[w]=0\ ;\ \ h_I^2[u,v]=0\ ;\ \ h_I^2[w,v]=0 \tag{23.0.2}$$

   satisfies $rk(u,v,w)<3$;

then h is steep in some neighborhood of I.

For $n=5$ we have

Theorem 6. Let $\mathcal{A}$ an open set of $\mathbb{R}^5$ and $h : \mathcal{A} \longrightarrow \mathbb{R}$ a smooth function. Let $I \in \mathcal{A}$ a point such that $\nabla h(I) \neq 0$.
If

1. h is four-jet non-degenerate at I;
2. for all $v \neq 0$ such that h is three-jet degenerate at I, any vector u solving

$$\begin{cases} h_I^1[u]=0\ ;\ \ h_I^2[u,v]=0\ ;\ \ h_I^2[u,u]h_I^4[v,v,v,v]=3(h_I^3[v,v,u])^2 \\ 15(h_I^3[v,v,u])^2 h_I^3[u,u,v]+h_I^5[v,v,v,v,v](h_I^2[u,u])^2 \\ \ =10h_I^4[v,v,v,u]h_I^3[u,v,v]h_I^2[u,u] \end{cases} \tag{23.0.3}$$

   satisfies $rk(u,v)<2$;

3. for all $v \neq 0$ such that h is three-jet degenerate at I, any couple of vectors (u,w) solving

$$\begin{cases} h_I^1[u]=0\ ;\ \ h_I^1[w]=0\ ;\ \ h_I^2[u,v]=0\ ;\ \ h_I^2[w,v]=0 \\ \{h_I^4[v,v,v,v]h_I^2[u,u]-6(h_I^3[u,v,v])^2\}\{h_I^2[w,w]h_I^2[u,u]-(h_I^2[u,w])^2\} \\ +12h_I^3[u,v,v]h_I^3[v,v,w]h_I^2[u,u]h_I^2[u,w]-6(h_I^3[u,v,v])^2(h_I^2[u,w])^2 \\ -6(h_I^3[v,v,w])^2(h_I^2[u,u])^2=0 \end{cases} \tag{23.0.4}$$

   satisfies $rk(u,v,w)<3$;

4. for all $v \neq 0$ such that h is two-jet degenerate at I, any triplet of vectors (u, w, x) solving

$$\begin{cases} h_I^1[u] = 0\ ;\ \ h_I^1[w] = 0\ ;\ \ h_I^1[x] = 0 \\ h_I^2[u, v] = 0\ ;\ \ h_I^2[w, v] = 0\ ;\ \ h_I^2[x, v] = 0 \end{cases} \tag{23.0.5}$$

satisfies $rk(u, w, x, v) < 4$;

then h is steep in some neighborhood of I.

Chapter 24

Examples

In this section, we test our results on some polynomial examples.

Example 1. The function

$$h(I) = h(I_1, I_2, I_3, I_4) = \frac{I_2^5}{5} + \frac{I_1^3}{3} - \frac{I_1^2}{2} + \frac{I_1 I_2}{2} - \frac{I_3^2}{2} - I_4 \tag{24.0.1}$$

is steep in a neighborhood of the origin $I = 0$.

Proof. We start be remarking that such function is three-jet and four-jet degenerate at the origin on those vectors $v \neq 0$ of the form

$$v := (v_1, v_2, v_3, v_4) = (0, v_2, 0, 0) \,, \tag{24.0.2}$$

so that neither Nekhoroshev explicit algebraic conditions for steepness, nor theorems in ref. [111] apply. However, the claim can be proven by applying Theorem 5. Indeed, it is easy to see that h is five-jet non-degenerate at $I = 0$. Moreover, system (23.0.1) reads

$$u_4 = 0 \,; \;\; u_1 v_1 + u_3 v_3 - \frac{1}{2} u_1 v_2 - \frac{1}{2} u_2 v_1 = 0 \,; \;\; v_2^5 (u_1^2 + u_3^2 - u_1 u_2)^2 = 0 \tag{24.0.3}$$

and, by taking expression (24.0.2) into account, one has that the only non-null solution is given by vectors of the kind $u = (0, u_2, 0, 0)$, which satisfy $rk(u, v) < 2$.
Finally, system (23.0.2) reads

$$\begin{cases} u_4 = w_4 = 0 \\ u_1 v_1 + u_3 v_3 - \frac{1}{2} u_1 v_2 - \frac{1}{2} u_2 v_1 = 0 \,; \;\; w_1 v_1 + w_3 v_3 - \frac{1}{2} w_1 v_2 - \frac{1}{2} w_2 v_1 = 0 \end{cases} \tag{24.0.4}$$

and, by taking expression (24.0.2) again into account, the only possible solutions are two families of vectors of the kind $u = (0, u_2, u_3, 0)$ and $w = (0, w_2, w_3, 0)$, which satisfy $rk(u, v, w) < 3$. Therefore, the hypotheses of Theorem 5 are fulfilled and the proof is concluded. □

Example 2. The function

$$h(I) = h(I_1, I_2, I_3, I_4, I_5) = \frac{I_4^4}{4} + \frac{I_5^4}{4} + \frac{I_3^3}{3} + \frac{I_3 I_2^2}{2} - \frac{I_1^2}{2} - \frac{I_3^2}{2} - \frac{I_5^2}{2} + I_3 I_4 + I_2 \quad (24.0.5)$$

is steep in a neighborhood of the origin $I = 0$.

Proof. We start by remarking that such function is two-jet degenerate at the origin on those vectors $z \neq 0$ of the form

$$z := (z_1, 0, z_3, z_4, z_5) \quad (24.0.6)$$

whose coordinates satisfy

$$z_1^2 + z_3^2 + z_5^2 - 2z_3 z_4 = 0 \, , \quad (24.0.7)$$

and three-jet degenerate on those vectors $v \neq 0$ of the kind

$$v := (0, 0, 0, v_4, 0) \, . \quad (24.0.8)$$

Therefore, Nekhoroshev's non-degeneracy conditions on the jets of order two and three are helpless in this case. Moreover, since such function has five degrees of freedom, the results in ref. [111] cannot be used (they only hold for $n = 2, 3, 4$). However, the claim can be proven by making use of Theorem 6. First, it is easy to see that h is four-jet non-degenerate at the origin. Moreover, by taking expression (24.0.8) into account, system (23.0.3) reads

$$u_2 = 0 \, ; \;\; u_3 = 0 \, ; \;\; u_1^2 + u_5^2 = 0 \, , \quad (24.0.9)$$

which is solved by vectors of the kind $u = (0, 0, 0, u_4, 0)$, that depend linearly on v. On the other hand, system (23.0.4) has the form

$$u_2 = 0 \, ; \;\; w_2 = 0 \, ; \;\; u_3 = 0 \, ; \;\; w_3 = 0 \, ; \;\; (u_1^2 + u_5^2)(u_1 w_5 - u_5 w_1)^2 = 0 \, , \quad (24.0.10)$$

where the particular form (24.0.8) of vector v has been taken into account once again. There are four possible cases

1. $u_1^2 + u_5^2 = 0$, which, by system (24.0.10), implies $u_1 = u_5 = 0$, so that vector u is of the kind $u = (0, 0, 0, u_4, 0)$, which is parallel to v;

2. $u_1 = 0, u_5 \neq 0$, which, by the last equation in (24.0.10), implies $w_1 = 0$, so that u and w have the form $u = (0, 0, 0, u_4, u_5)$ and $w = (0, 0, 0, w_4, w_5)$, so that $rk(u, v, w) < 3$;

3. $u_1 \neq 0, u_5 = 0$ which is similar to the previous point and yields u, w of the kind $u = (u_1, 0, 0, u_4, 0)$, $w = (w_1, 0, 0, w_4, 0)$, so that $rk(u, v, w) < 3$;

4. $u_1 \neq 0, u_5 \neq 0$ which, by system (24.0.10), yields $u_5 w_1 - u_1 w_5 = 0$. Therefore, one has
$$\det \begin{pmatrix} 0 & 0 & 0 & v_4 & 0 \\ u_1 & 0 & 0 & u_4 & u_5 \\ w_1 & 0 & 0 & w_4 & w_5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = v_4(u_5 w_1 - u_1 w_5) = 0$$
so that, since
$$rk \begin{pmatrix} 0 & 0 & 0 & v_4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = 3\,,$$
one must have $rk(u, w) < 2$ and, consequently, $rk(u, v, w) < 3$.

Finally, system (23.0.5) reads
$$\begin{cases} u_2 = 0\,; \quad w_2 = 0\,; \quad x_2 = 0 \\ u_1 z_1 + u_3 z_3 + u_5 z_5 - u_3 z_4 - u_4 z_3 = 0 \\ w_1 z_1 + w_3 z_3 + w_5 z_5 - w_3 z_4 - w_4 z_3 = 0 \\ x_1 z_1 + x_3 z_3 + x_5 z_5 - x_3 z_4 - x_4 z_3 = 0 \end{cases} \tag{24.0.11}$$
which means that the vectors u, w, x belong to the three-dimensional subspace orthogonal to $Span\{(0, 1, 0, 0, 0), (z_1, 0, z_3 - z_4, -z_3, z_5)\}$. By looking at expressions (24.0.6) and (24.0.7), we see that vector z belongs to the same subspace, so that $rk(u, w, x, z) < 4$. The hypotheses of theorem 6 are therefore fulfilled and the proof is concluded. □

Chapter 25

The steepness property in the space of jets

We start by recalling Nekhoroshev's abstract construction (see ref. [96]) of the sets $\sigma_n^r(I)$ in Theorem 2, with $r, n \geq 2$, whose closures contain all non steep functions with non-zero gradient at the point I.

Definition 3. Take two integers $r, n \geq 2$ and define $\beta_m := \frac{\bar{\alpha}_m + 3}{2}$, with $\bar{\alpha}_m$ as in Theorem 2.
$\sigma_n^r(I) \subset \mathcal{P}_I(r, n)$ is the set containing the r-jets of smooth functions h such that

1. $\nabla h(I) \neq 0$
2. There exists an m-dimensional subspace Λ_m^I orthogonal to $\nabla h(I)$ and a curve $\gamma : \mathbb{R} \longrightarrow \Lambda_m^I$ of the form

$$\gamma(t) := \begin{cases} x_1(t) = t \\ x_i(t) = \sum_{j=1}^{\beta_m - 1} b_{ij} t^j \ , \quad i \in \{2, ..., m\} \ , \quad b_{ij} \in \mathbb{R} \end{cases} , \tag{25.0.1}$$

   such that the restriction of the gradient of h to $\gamma(t)$ has a zero of order not smaller than $\beta_m - 1$ at $t = 0$:

$$\left. \frac{d^p (\nabla h|_{\Lambda_m^I})|_{\gamma(t)}}{dt^p} \right|_{t=0} = 0 \ , \quad p \in \{1, 2, ..., \beta_m - 1\} \ . \tag{25.0.2}$$

Remark. The reader might wonder why the value $\beta_m = (\alpha_m + 3)/2$ was chosen in the definition of $\sigma_n^r(I)$. Infact, in his first work on the genericity of steepness, Nekhoroshev proves that, for any fixed $\beta_m \in \mathbb{N}$, $\beta_m > 1$, any polynomial $P \in \mathcal{P}_I(r, n) \backslash \sigma_n^r(I)$ is steep on the subspace Λ_m^I with indices $\alpha_m = 2(\beta_m - 1) - 1$, hence $\beta_m = (\alpha_m + 3)/2$ (see Theorem C and Lemma 7.2.2 in ref. [94]).

With this definition, we can write down the algebraic conditions that the r-jet $P_I(h,r,n)$ of a smooth function h must satisfy in some m-dimensional subspace Λ^I_m in order to belong to $\sigma^r_n(I)$. For fixed m, these can be gathered in a system $\Xi_m(h,I,n)$ composed of four subsystems $\xi_{m,l}$, with $l=1,2,3,4$,

$$\Xi_m(h,I,n) := \begin{cases} \xi_{m,1}(h)\ ;\ \ \xi_{m,2}(h,A^i) \\ \xi_{m,3}(h,A^i)\ ;\ \ \xi_{m,4}(h,A^i,b_{ij}) \end{cases} ,$$

where $i\in\{1,...,m\}$, $j\in\{1,...,\beta_m-1\}$, the A^i are linearly independent vectors (with origin at I) which constitute a basis for Λ^I_m and the coefficients b_{ij} are real parameters defining a curve $\gamma(t)$ as in (25.0.1). One has that

1. $\xi_1(h)$ imposes $\nabla h(I)\neq 0$;
2. $\xi_2(h,A^i)$ imposes the vectors $A^1,...,A^m$ to be linearly independent,
$$rk[A^1,...,A^m]=m\ ;$$
3. $\xi_3(h,A^i)$ imposes the vectors $A^1,...,A^m$ to be orthogonal to $\nabla h(I)$,
$$h^1_I[A^1]=0\ ;\ \ ...\ ;\ \ h^1_I[A^m]=0\ ;$$
4. $\xi_4(h,A^i,b_{ij})$ is a system of $m(\beta_m-1)$ equations obtained as follows. We denote by $x_1,...,x_m$ the coordinates for Λ^I_m with respect to the basis $A^1,...,A^m$. By construction, such coordinates are null at I. Then, we consider the Taylor polynomial of $h|_{\Lambda^I_m}$ at I up to order β_m, namely
$$P^{\beta_m}_n(x) := \sum_{i=1}^m h^1_I[A^i]x_i + \frac{1}{2}\sum_{i,j=1}^m h^2_I[A^i,A^j]x_ix_j + ... + \frac{1}{\beta_m!}\underbrace{\sum_{i,j,k,...,l=1}^m}_{\beta_m \text{ terms}} h^{\beta_m-1}_I[A^i,A^j,A^k,...,A^l]x_ix_jx_k...x_l\ . \tag{25.0.3}$$
Condition (25.0.2) can now be imposed by considering the gradient $\nabla P^{\beta_m}_n(x)$, by injecting expression (25.0.1) in each of its m components and by requiring that the β_m-1 coefficients of the resulting polynomial in t are null. One thus obtains $m(\beta_m-1)$ equations.

For fixed m, $\Xi_m(h,I,n)$ is said to be solvable for a given h at I if there exist a basis $A^1,...,A^m$ and real parameters b_{ij} that verify it. $P_I(h,r,n)$ belongs to $\sigma^r_n(I)$ if at least one of the systems $\Xi_m(h,I,n)$, with $m\in\{1,...,n-1\}$, is solvable for h.

Indeed, following Theorem 2, in the sequel we will try to consider the closure of the algebraic conditions defining $\sigma_n^5(I)$ and, when this turns out to be too complicated, we will choose suitable closed sets containing $\sigma_n^5(I)$, with $n = 2, 3, 4, 5$. We will not deal with the case $n \geq 6$ since in such situation the conditions we find yield sets of steep functions which are smaller than those yielded by the three-jet non degeneracy condition, as it was already pointed out in ref. [111].

Chapter 26

Proofs

For the sake of simplicity, from now on we drop the subscript I in h_I referring to the point where the considered jet is calculated. Moreover, we denote by $\Pi_{\Lambda_m^I}$ the projection onto an m-dimensional linear affine subspace Λ_m^I orthogonal to the gradient. We start by stating the following simple lemma, which will turn out to be useful when trying to prove the closedness of the sets which we shall consider in the sequel, namely

Lemma 1. Let E be a metric space, K a compact subset of some metric space and Δ a closed subset of $E \times K$. Then, the projection of Δ on E, denoted by $\Pi_E(\Delta)$, is closed.

Proof. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in $\Pi_E(\Delta)$ converging to a point $\bar{p}$ and $\{k_n\}_{n\in\mathbb{N}}$ a sequence in K satisying $(p_n, k_n) \in \Delta$. Since K is a compact subset of some metric space, one can extract a subsequence $\{k_{n_l}\}_{l\in\mathbb{N}}$ converging to a point $\bar{k} \in K$. Hence, the sequence $\{(p_{n_l}, k_{n_l})\}_{l\in\mathbb{N}}$ in Δ converges to $(\bar{p}, \bar{k}) \in \Delta$, since Δ is closed. This implies that $\bar{p}$ belongs to $\Pi_E(\Delta)$, which is therefore closed. □

The following sets will turn out to be particularly useful in the sequel.

Definition 4. For $n = 2, 3, 4$, we denote by $\Psi_1^*(n) \subset \mathcal{P}_I(5, n)$ the set of those jets of order five which satisfy the five-jet degeneracy condition. Similarly, for $n = 5$ we denote by $\Psi_1^*(5) \subset \mathcal{P}_I(5, 5)$ the set of those jets of order five which are four-jet degenerate. Moreover, for $n = 1, 2, 3, 4, 5$, we indicate by $\Psi_1(n) \subset \mathcal{P}_I(5, n)$ the intersection between $\Psi_1^*(n)$ and the set containing those jets corresponding to functions having non-zero gradient at I.

In particular, by Lemma 4.1 in ref. [111] one has

Lemma 2. For $n = 2, 3, 4, 5$, the set $\Psi_1^*(n)$ is closed and it coincides with the closure of $\Psi_1(n)$.

With this setup, we are now ready to give the proofs of Theorems 3-6.

26.1 Proof of Theorem 3 (n=2)

Proof. We assume the hypotheses of Theorem 3. Since we are in a domain of $\mathbb{R}^2$, the only possible dimension for a subspace orthogonal to the gradient is $m = 1$. For $n = 2$ and $r = 5$ we have $\beta_1 = 5$. Now, we build the set $\sigma_2^5(I)$ by following the strategy described by Nekhoroshev in [96] and which we recalled in Theorem 2 and Definition 3. First, we consider the Taylor polynomial of the restriction of the function h to the subspace Λ_1^I up to order $\beta_1 = 5$:

$$\begin{aligned} P_2^5(x) =& h^1[v]x + \frac{1}{2}h^2[v,v]x^2 + \frac{1}{6}h^3[v,v,v]x^3 \\ &+ \frac{1}{24}h^4[v,v,v,v]x^4 + \frac{1}{120}h^5[v,v,v,v,v]x^5 \ , \end{aligned} \tag{26.1.1}$$

where v is a non-null vector orthogonal to the gradient. Then, we calculate $\nabla P_5^2(x)$ and we consider its restriction to the curve $x(t) = t$. By setting all the coefficients of such polynomial to be equal to zero we obtain the subsystem $\xi_4(h, v)$ described in the previous section, so that system $\Xi_1(h, I)$ reads

$$\begin{cases} \nabla h(I) \neq 0\ ;\ \ v \neq 0\ ;\ \ h^1[v] = 0\ ;\ \ h^2[v,v] = 0 \\ h^3[v,v,v] = 0\ ;\ \ h^4[v,v,v,v] = 0\ ;\ \ h^5[v,v,v,v,v] = 0 \end{cases} . \tag{26.1.2}$$

Since this is the only system we can consider in this case, we have that the set $\sigma_2^5(I)$ coincides with the one defined by $\Xi_1(h, I)$ which, in turn, is equal to $\Psi_1(2)$ by Definition 4. Theorem 3 then follows from Lemma 2 and Theorem 2. □

26.2 Proof of the case $n = 3$

The goal is to prove Theorem 4. Analogously to the case $n = 2$, we give some suitable definitions.

Definition 5. We denote by $\Psi_2(3)$ the set in the space of 5-jets of smooth functions h of three variables such that there exist two linearly independent vectors u, v and two real parameters α, β satisfying

$$\begin{cases} \nabla h(I) \neq 0\ ;\ \ h^1[u] = h^1[v] = \Pi_{\Lambda_2^I} h^2[v,\cdot] = \Pi_{\Lambda_2^I}\left(2\alpha h_I[u,\cdot] + h^3[v,v,\cdot]\right) = 0 \\ \Pi_{\Lambda_2^I}(6\beta h^2[u,\cdot] + 6\alpha h^3[u,v,\cdot] + h_I^4[v,v,v,\cdot]) = 0\ . \end{cases} \tag{26.2.1}$$

Definition 6. We denote by $\Psi_2^*(3)$ the set in the space of 5-jets of smooth functions h of three variables such that there exists two linearly independent vectors u, v satisfying

$$\begin{cases} h^1[u] = h^1[v] = h^2[v,v] = h^2[v,u] = h^3[v,v,v] = 0 \\ h^2[u,u]h^4[v,v,v,v] = 3(h^3[v,v,u])^2 \end{cases} . \tag{26.2.2}$$

The following result holds true

Lemma 3. The set $\Psi_2^*(3)$ is closed and contains the closure of $\Psi_2(3)$.

Proof. We notice that all equations in (26.2.2) are homogeneous in u and v, so that without any loss of generality we can consider $(u, v) \in \mathbb{S}^2 \times \mathbb{S}^2$. Moreover, still without any loss of generality we can assume $u \cdot v = 0$, since it is easy to see that the component of u which is parallel to v yields a null contribution to the system in (26.2.2). Then, system (26.2.2) defines an algebraic closed set in $\mathcal{P}_I(5,3) \times \mathbb{S}^2 \times \mathbb{S}^2$, whose projection onto $\mathcal{P}_I(5,3)$ is $\Psi_2^*(3)$. Hence $\Psi_2^*(3)$ is closed by Lemma 1. In order to prove inclusion, we write the system defining $\Psi_2(3)$ in its less compact form

$$\begin{cases} \nabla h(I) \neq 0 \,; \quad h^1[u] = 0 \,; \quad h^1[v] = 0 \,; \quad h^2[v,v] = 0 \\ h^2[v,u] = 0 \,; \quad h^3[v,v,v] = 0 \,; \quad 6\alpha h^3[u,v,v] + h^4[v,v,v,v] = 0 \\ 2\alpha h^2[u,u] + h^3[u,v,v] = 0 \\ 6\beta h^2[u,u] + 6\alpha h^3[u,u,v] + h^4[v,v,v,u] = 0 \end{cases} . \tag{26.2.3}$$

By applying Gauss elimination method to the last two equations and by subtracting one to another, one can get rid of parameter α and obtains

$$3(h^3[u,v,v])^2 = h^4[v,v,v,v]h^2[u,u] \,.$$

Then, by discarding the last equation and the inequality on the gradient of h, one reduces to the system defining the set $\Psi_2^*(3)$. Therefore the inclusion $\Psi_2^*(3) \supset \Psi_2(3)$ holds. Since $\Psi_2^*(3)$ is closed, one has $\Psi_2^*(3) \supset \bar{\Psi}_2(3)$ and the statement is thus proven. □

We remark that we considered set $\Psi_2^*(3)$ since $\Psi_2(3)$ is not closed, as we show in the following

Example 3. For $k \in \mathbb{N}$, consider the sequence of polynomial functions

$$h_k(I_1, I_2, I_3) = \frac{3}{2}\frac{I_1^4 + I_2^4}{4!} - \frac{I_3^4}{4!k} - \frac{I_2 I_1^2}{2k} + \frac{I_2^2}{2k^2} + I_3 \,,$$

converging to $h(I_1, I_2, I_3) = \frac{3}{2}\frac{I_1^4 + I_2^4}{4!} + I_3$. At the origin, the jet $P(h_k, 5, 3)$ associated to h_k belongs to the set $\Psi_2(3)$ for all k, but the jet $P(h, 5, 3)$ associated to the limit function does not.

Proof. For fixed $k \in \mathbb{N}$, set $\alpha_k := \frac{k}{2}$, $\beta_k := \frac{k^2}{2}$ and the vectors $u = (0,1,0)$, $v = (1,0,0)$. It is straightforward to see that $P(h_k, 5, 3) \in \Psi_2(3)$ at the origin, with such choice of vectors and parameters. However, the limit function h is weakly-convex at the origin and, as the reader can easily verify, it does not fulfill system (26.2.3) for any non-null vector v. □

We are now ready to write the proof of Theorem 4.

Proof. We assume the hypotheses of Theorem 4. Since we are in a domain of $\mathbb{R}^3$, m can be equal to 1 or 2. For $n = 3$ and $r = 5$ we have $\beta_1 = 5$ and $\beta_2 = 4$.
For $m = 1$, by following the same construction as in the case $n = 2$, we have the same expression of (26.1.2) for $\Xi_1(h, I, 3)$.
In order to build up system $\Xi_2(h, I, 3)$, we follow the usual strategy described by Nekhoroshev in [96] and we consider the Taylor polynomial of the restriction of the function h to the subspace Λ_2^I up to order $\beta_2 = 4$. By calculating $\nabla P_4^3(x) = (\partial_{x_1} P_4^3(x), \partial_{x_2} P_4^3(x))$ along the curve

$$x_1 = t\ ;\quad x_2 = b_{21}t + b_{22}t^2 + b_{23}t^3$$

and by setting equal to zero all the coefficients of the resulting polynomial in t up to order $\beta_2 - 1 = 3$, one has that

1. The linear terms yield $\Pi_{\Lambda_2^I} h^2[A^1 + b_{21}A^2, \cdot] = 0$;
2. The quadratic terms yield

$$\Pi_{\Lambda_2^I}\left(2b_{22}h^2[A^2, \cdot] + h^3[A^1 + b_{21}A^2, A^1 + b_{21}A^2, \cdot]\right) = 0\ ; \tag{26.2.4}$$

3. Finally, the cubic terms yield

$$\begin{aligned}\Pi_{\Lambda_2^I}(&6b_{23}h^2[A^2, \cdot] + 6b_{22}h^3[A^2, A^1 + b_{21}A^2, \cdot]\\ &+h^4[A^1 + b_{21}A^2, A^1 + b_{21}A^2, A^1 + b_{21}A^2, \cdot]) = 0\ ;\end{aligned} \tag{26.2.5}$$

where A^1, A^2 are a basis for Λ_2^I.
Thus, system $\Xi_2(h, I, 3)$ takes the form

$$\begin{cases}\nabla h(I) \neq 0\ ;\ \ h^1[u] = h^1[v] = \Pi_{\Lambda_2^I} h^2[v, \cdot] = \Pi_{\Lambda_2^I}\left(2\alpha h^2[u, \cdot] + h^3[v, v, \cdot]\right) = 0\\ \Pi_{\Lambda_2^I}(6\beta h^2[u, \cdot] + 6\alpha h^3[u, v, \cdot] + h^4[v, v, v, \cdot]) = 0\ ,\end{cases} \tag{26.2.6}$$

with $u := A^2, v := A^1 + b_{21}A^2$ two linearly independent vectors and $\alpha := b_{22}, \beta := b_{23}$ two real parameters. With the help of Definitions 4 and 5 we see that $\sigma_3^5(I) = \Psi_1(3) \cup \Psi_2(3)$. As a consequence of Lemmas 2 and 3 and of Theorem 2 one has $\Sigma_3^5(I) = \bar{\sigma}_3^5(I) \subset \Psi_1^*(3) \cup \Psi_2^*(3)$ and Theorem 4 follows. □

26.3 Proof of the case $n = 4$

The goal is to prove Theorem 5. We start with the usual definitions

Definition 7. We denote by $\Psi_2(4)$ the set in the space of 5-jets of smooth functions h of four variables such that there exist two linearly independent vectors u, v and three real parameters α, β, γ satisfying

$$\begin{cases} \nabla h(I) \neq 0\,; \quad h^1[u] = h^1[v] = \Pi_{\Lambda_2^I} h^2[v,\cdot] = \Pi_{\Lambda_2^I}\left(2\alpha h^2[u,\cdot] + h^3[v,v,\cdot]\right) = 0 \\ \Pi_{\Lambda_2^I}(6\beta h^2[u,\cdot] + 6\alpha h^3[u,v,\cdot] + h^4[v,v,v,\cdot]) = 0 \\ \Pi_{\Lambda_2^I}(24\gamma h^2[u,\cdot] + 24\beta h^3[u,v,\cdot] \\ +12\alpha^2 h^3[u,u,\cdot] + 12\alpha h^4[v,v,u,\cdot] + h^5[v,v,v,v,\cdot]) = 0 \end{cases} . \tag{26.3.1}$$

Definition 8. We denote by $\Psi_2^*(4)$ the set in the space of 5-jets of smooth functions h of four variables such that there exist two linearly independent vectors u, v satisfying

$$\begin{cases} h^1[u] = h^1[v] = h^2[v,v] = h^2[v,u] = h^3[v,v,v] = 0 \\ h^2[u,u]h^4[v,v,v,v] = 3(h^3[v,v,u])^2 \\ 15(h^3[v,v,u])^2 h^3[u,u,v] + h^5[v,v,v,v,v](h^2[u,u])^2 \\ \;= 10h^4[v,v,v,u]h^3[u,v,v]h^2[u,u] \end{cases} . \tag{26.3.2}$$

Definition 9. We denote by $\Psi_3(4)$ the set in the space of 5-jets of smooth functions h of four variables such that there exist three linearly independent vectors u, v, w and two real parameters α, β satisfying

$$\begin{cases} \nabla h(I) \neq 0\,; \quad h^1[u] = h^1[v] = h^1[w] = \Pi_{\Lambda_3^I} h^2[v,\cdot] = 0 \\ \Pi_{\Lambda_3^I}(2h^2[\alpha u + \beta w,\cdot] + h^3[v,v,\cdot]) = 0 \end{cases} . \tag{26.3.3}$$

Definition 10. We denote by $\Psi_3^*(4)$ the set in the space of 5-jets of smooth functions h of four variables such that there exist three linearly independent vectors u, v, w satisfying

$$h^1[u] = h^1[v] = h^1[w] = h^2[v,v] = h^2[v,u] = h^2[v,w] = 0\,; \quad h^3[v,v,v] = 0\,. \tag{26.3.4}$$

With these definitions, we have the following

Lemma 4. The sets $\Psi_2^*(4), \Psi_3^*(4)$ are closed and one also has the inclusions $\Psi_2^*(4) \supseteq \bar{\Psi}_2(4), \Psi_3^*(4) \supseteq \bar{\Psi}_3(4)$.

Proof. The proof is similar to that of Theorem 3: without any loss of generality, one can always choose the vectors to be perpendicular and unitary, so that systems (26.3.2) and (26.3.4) define algebraic closed sets in $\mathcal{P}_I(5,n) \times \mathbb{S}^2 \times \mathbb{S}^2$ and $\mathcal{P}_I(5,n) \times \mathbb{S}^3 \times \mathbb{S}^3$, whose projections onto $\mathcal{P}_I(5,n)$ are $\Psi_2^*(4)$ and $\Psi_3^*(4)$, which are therefore closed thanks to Lemma 1. As for the inclusions, the relation $\Psi_3(4) \subset \Psi_3^*(4)$ is immediate from definition 9, once one projects the equations on the basis u, v, w and compares the system to the one in definition 10.

In order to prove that $\Psi_2(4) \subset \Psi_2^*(4)$, we consider system (26.3.1) defining $\Psi_2(4)$ in its most explicit form

$$\begin{cases} \nabla h(I) \neq 0\,; \quad h^1[u] = h^1[v] = 0\,; \quad h^2[v,v] = h^2[v,u] = 0 \\ h^3[v,v,v] = 0\,; \quad 2\alpha h^2[u,u] + h^3[v,v,u] = 0 \\ 6\alpha h^3[u,v,v] + h^4[v,v,v,v] = 0 \\ 6\beta h^2[u,u] + 6\alpha h^3[u,u,v] + h^4[v,v,v,u] = 0 \\ 24\beta h^3[u,v,v] + 12\alpha^2 h^3[u,u,v] + 12\alpha h^4[v,v,v,u] + h^5[v,v,v,v,v] = 0 \\ 24\gamma h^2[u,u] + 24\beta h^3[u,u,v] \\ +12\alpha^2 h^3[u,u,u] + 12\alpha h^4[v,v,u,u] + h^5[v,v,v,v,u] = 0 \end{cases} . \tag{26.3.5}$$

Applying Gauss elimination method in order to get rid of parameters α, β and discarding the last equation containing parameter γ yields system (26.3.2) defining $\Psi_2^*(4)$. Therefore, one has $\Psi_2(4) \subset \Psi_2^*(4)$. Since $\Psi_2^*(4)$ and $\Psi_3^*(4)$ are closed, one finally obtains $\Psi_2^*(4) \supseteq \bar{\Psi}_2(4)$, $\Psi_3^*(4) \supseteq \bar{\Psi}_3(4)$. □

With this setup, we are ready to prove Theorem 5.

Proof. Since we work in a domain of $\mathbb{R}^4$, m can be equal to $1, 2$ or 3. For $n = 4$ and $r = 5$ we have $\beta_1 = \beta_2 = 5$ and $\beta_3 = 3$.
For $m = 1$, we follow the same construction as in the cases $n = 2, 3$ and system $\Xi_1(h, I, 4)$ defines a set $\Psi_1(4)$ whose closure concides with $\Psi_1^*(4)$. In order to build up system $\Xi_2(h, I, 4)$, we follow once again the construction in [96] and we consider the Taylor polynomial of the restriction of the function h to a subspace Λ_2^I up to order $\beta_2 = 5$. By calculating $\nabla P_5^4(x) = (\partial_{x_1} P_5^4(x), \partial_{x_2} P_5^4(x))$ along the curve

$$x_1 = t\,; \quad x_2 = b_{21}t + b_{22}t^2 + b_{23}t^3 + b_{24}t^4$$

and by setting equal to zero all the coefficients of the resulting polynomial in t up to order $\beta_2 - 1 = 4$, one has that

1. The linear terms yield $\Pi_{\Lambda_2^I} h^2[A^1 + b_{21}A^2, \cdot] = 0$;

2. The quadratic terms yield

$$\Pi_{\Lambda_2^I}\left(2b_{22}h^2[A^2,\cdot] + h^3[A^1 + b_{21}A^2, A^1 + b_{21}A^2, \cdot]\right) = 0\,; \tag{26.3.6}$$

3. The cubic terms yield

$$\begin{aligned} \Pi_{\Lambda_2^I}(&6b_{23}h^2[A^2,\cdot] + 6b_{22}h^3[A^2, A^1 + b_{21}A^2, \cdot] \\ &+h^4[A^1 + b_{21}A^2, A^1 + b_{21}A^2, A^1 + b_{21}A^2, \cdot]) = 0\,; \end{aligned} \tag{26.3.7}$$

4. The quartic terms yield

$$\Pi_{\Lambda_2}(24b_{24}h^2[A^2,\cdot] + 24b_{23}h^3[A^1+b_{21}A^2, A^2,\cdot] + 12b_{22}^2h^3[A^2,A^2,\cdot] \\ +12b_{22}h^4[A^1+b_{21}A^2, A^1+b_{21}A^2, A^2,\cdot] \\ +h^5[A^1+b_{21}A^2, A^1+b_{21}A^2, A^1+b_{21}A^2, A^1+b_{21}A^2,\cdot]) = 0\,; \tag{26.3.8}$$

where A^1, A^2 are a basis for Λ_2^I. Thus, by following the same strategy as in the previous section, $\Xi_2(h,I)$ has the form

$$\begin{cases} \nabla h(I) \neq 0\,;\;\; h^1[u] = h^1[v] = 0\,;\;\; \Pi_{\Lambda_2^I} h^2[v,\cdot] = 0 \\ \Pi_{\Lambda_2^I}\left(2\alpha h^2[u,\cdot] + h^3[v,v,\cdot]\right) = 0 \\ \Pi_{\Lambda_2^I}(6\beta h^2[u,\cdot] + 6\alpha h^3[u,v,\cdot] + h^4[v,v,v,\cdot]) = 0 \\ \Pi_{\Lambda_2}(24\gamma h^2[u,\cdot] + 24\beta h^3[v,u,\cdot] + 12\alpha^2 h^3[u,u,\cdot] \\ +12\alpha h^4[v,v,u,\cdot] + h^5[v,v,v,v,\cdot]) = 0\,. \end{cases} \tag{26.3.9}$$

with $u := A^2, v := A^1 + b_{21}A^2$ two linearly independent vectors and $\alpha := b_{22}, \beta := b_{23}, \gamma := b_{24}$ three real parameters. Finally, we construct system $\Xi_3(h,I)$. We consider the Taylor polynomial $P_5^4(x)$ of the restriction of the function h to the subspace Λ_3^I, up to order $\beta_3 = 3$. By calculating $\nabla P_5^4(x) = (\partial_{x_1}P_5^4(x), \partial_{x_2}P_5^4(x), \partial_{x_3}P_5^4(x))$ along the curve

$$x_1 = t\,;\;\; x_2 = b_{21}t + b_{22}t^2\,;\;\; x_3 = b_{31}t + b_{32}t^2$$

and by setting equal to zero all the coefficients of the resulting polynomial in t up to order $\beta_3 - 1 = 2$, one has that

1. The linear terms yield $\Pi_{\Lambda_3^I}(h^2[A^1 + b_{21}A^2 + b_{31}A^3,\cdot]) = 0\,;$

2. The quadratic terms yield

$$\Pi_{\Lambda_3^I}(h^2[2b_{22}A^2 + 2b_{32}A^3,\cdot] \\ +h^3[A^1+b_{21}A^2+b_{31}A^3, A^1+b_{21}A^2+b_{31}A^3,\cdot]) = 0\,; \tag{26.3.10}$$

   where A_1, A_2, A_3 are a basis for Λ_3^I.

Thus, by following the same strategy as in the previous section, $\Xi_3(h,I)$ reads

$$\begin{cases} \nabla h(I) \neq 0\,;\;\; h^1[u] = 0\,;\;\; h^1[v] = 0\,;\;\; h^1[w] = 0 \\ \Pi_{\Lambda_3^I} h^2[v,\cdot] = 0\,;\;\; \Pi_{\Lambda_3^I}\left(2h^2[\alpha u + \beta w,\cdot] + h^3[v,v,\cdot]\right) = 0 \end{cases}. \tag{26.3.11}$$

with $u := A^2, v := A^1 + b_{21}A^2 + b_{31}A^3$, $w := A^3$ three linearly independent vectors and $\alpha := b_{22}, \beta := b_{32}$ two real parameters. With the help of Definitions 4, 7 and 9, we see that $\sigma_4^5(I) = \Psi_1(4) \cup \Psi_2(4) \cup \Psi_3(4)$, so that, as a consequence of Lemma 4 and of Theorem 2, one has $\Sigma_4^5(I) = \bar{\sigma}_4^5(I) \subset \Psi_1^*(4) \cup \Psi_2^*(4) \cup \Psi_3^*(4)$, which, together with Theorem 2 once again, implies Theorem 5. □

26.4 Proof of the case $n = 5$

The goal consists in proving Theorem 6. We start with the usual definitions

Definition 11. We denote by $\Psi_2(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist two linearly independent vectors u, v and three real parameters α, β, γ satisfying

$$\begin{cases} \nabla h(I) \neq 0\,; \;\; h^1[u] = h^1[v] = \Pi_{\Lambda_2^I} h^2[v,\cdot] = \Pi_{\Lambda_2^I}\left(2\alpha h^2[u,\cdot] + h^3[v,v,\cdot]\right) = 0 \\ \Pi_{\Lambda_2^I}(6\beta h^2[u,\cdot] + 6\alpha h^3[u,v,\cdot] + h^4[v,v,v,\cdot]) = 0 \\ \Pi_{\Lambda_2^I}(24\gamma h^2[u,\cdot] + 24\beta h^3[u,v,\cdot] \\ +12\alpha^2 h^3[u,u,\cdot] + 12\alpha h^4[v,v,u,\cdot] + h^5[v,v,v,v,\cdot]) = 0 \end{cases} . \tag{26.4.1}$$

Definition 12. We denote by $\Psi_2^*(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist two linearly independent vectors u, v satisfying

$$\begin{cases} h^1[u] = h^1[v] = h^2[v,v] = h^2[v,u] = h^3[v,v,v] = 0 \\ h^2[u,u]h^4[v,v,v,v] = 3(h^3[v,v,u])^2 \\ 15(h^3[v,v,u])^2 h^3[u,u,v] + h^5[v,v,v,v,v](h^2[u,u])^2 \\ \;= 10h^4[v,v,v,u]h^3[u,v,v]h^2[u,u] \end{cases} . \tag{26.4.2}$$

Definition 13. We denote by $\Psi_3(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist three linearly independent vectors u, v, w and four real parameters $\alpha, \beta, \gamma, \delta$ satisfying

$$\begin{cases} \nabla h(I) \neq 0\,; \;\; h^1[u] = h^1[v] = h^1[w] = \Pi_{\Lambda_3^I} h^2[v,\cdot] = 0 \\ \Pi_{\Lambda_3^I}(h^2[\alpha u + \beta w,\cdot] + h^3[v,v,\cdot]) = 0 \\ \Pi_{\Lambda_3^I}(6h^2[\gamma u + \delta w,\cdot] + 6h^3[\alpha u + \beta w, v,\cdot] + h^4[v,v,v,\cdot]) = 0 \end{cases} . \tag{26.4.3}$$

Definition 14. We denote by $\Psi_3^*(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist three linearly independent vectors u, v, w satisfying

$$\begin{cases} h^1[u] = h^1[v] = h^1[w] = h^2[v,v] = h^2[v,u] = h^2[v,w] = h^3[v,v,v] = 0 \\ 12h^3[u,v,v]h^3[v,v,w]h^2[u,u]h^2[u,w] \\ \;-6(h^3[u,v,v])^2(h^2[u,w])^2 - 6(h^3[v,v,w])^2(h^2[u,u])^2 \\ \;+\{h^4[v,v,v,v]h^2[u,u] - 6(h^3[u,v,v])^2\}\{h^2[w,w]h^2[u,u] - (h^2[u,w])^2\} = 0 \end{cases} . \tag{26.4.4}$$

Definition 15. We denote by $\Psi_4(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist four linearly independent vectors u, v, w, x satisfying

$$\nabla h(I) \neq 0\,;\quad h^1[u] = h^1[v] = h^1[w] = h^1[x] = 0\,;\quad \Pi_{\Lambda_4^I} h^2[v,\cdot] = 0\,. \tag{26.4.5}$$

Definition 16. We denote by $\Psi_4^*(5)$ the set in the space of 5-jets of smooth functions h of five variables such that there exist four linearly independent vectors u, v, w, x satisfying

$$h^1[u] = h^1[v] = h^1[w] = h^1[x] = 0\,;\quad \Pi_{\Lambda_4^I} h^2[v,\cdot] = 0\,. \tag{26.4.6}$$

We have the following result:

Lemma 5. The sets $\Psi_2^*(5), \Psi_3^*(5), \Psi_4^*(5)$ are closed and the following inclusions hold: $\Psi_2^*(5) \supset \bar{\Psi}_2(5), \Psi_3^*(5) \supset \bar{\Psi}_3(5), \Psi_4^*(5) \supset \bar{\Psi}_4(5)$.

Proof. Closure of the three sets $\Psi_2^*(5), \Psi_3^*(5), \Psi_4^*(5)$ is proven exactly in the same way as in the previous paragraphs, with the help of Lemma 1. The proof of the inclusion $\Psi_2^*(5) \supset \bar{\Psi}_2(5)$ is identic to the one given in Lemma 4 for the inclusion $\Psi_2^*(4) \supset \bar{\Psi}_2(4)$. Inclusion $\Psi_4^*(5) \supset \bar{\Psi}_4(5)$ is immediate when considering the definitions of $\Psi_4(5)$ and $\Psi_4^*(5)$ and the closure of the latter. The only non-trivial inclusion is thus $\Psi_3^*(5) \supset \bar{\Psi}_3(5)$. In order to prove it, we rewrite the system defining $\Psi_3(5)$ in its less synthetic form

$$\begin{cases} \nabla h(I) \neq 0\,;\quad h^1[u] = h^1[v] = h^1[w] = h^2[v,v] = h^2[v,u] = h^2[v,w] = 0 \\ h^3[v,v,v] = 0\,;\quad h^2[\alpha u + \beta w, u] + h^3[v,v,u] = 0 \\ h^2[\alpha u + \beta w, w] + h^3[v,v,w] = 0 \\ 6h^2[\gamma u + \delta w, u] + 6h^3[\alpha u + \beta w, v, u] + h^4[v,v,v,u] = 0 \\ 6h^2[\gamma u + \delta w, w] + 6h^3[\alpha u + \beta w, v, w] + h^4[v,v,v,w] = 0 \\ 6h^3[\alpha u + \beta w, v, v] + h^4[v,v,v,v] = 0 \end{cases} . \tag{26.4.7}$$

Once again, Gauss elimination method can be used in order to get rid of parameters α and β. Then, by discarding the first inequality and the two equations containing γ, δ in system (26.4.7) defining $\Psi_3(5)$, one obtains the system in Definition 10, which determines $\Psi_3^*(5)$. Therefore, one has $\Psi_3^*(5) \supset \Psi_3(5)$ and $\Psi_3^*(5) \supset \bar{\Psi}_3(5)$ since $\Psi_3^*(5)$ is closed. □

With this background, we are ready to prove Theorem 6.

Proof. Since we work in a domain of $\mathbb{R}^5$, m can be equal to $1, 2, 3$ or 4. For $n = 5$ and $r = 5$ we have $\beta_1 = 4, \beta_2 = 5, \beta_3 = 4$ and $\beta_4 = 2$.
For $m = 1$, by following the same construction as in the cases $n = 2, 3, 4$, we find the following expression for $\Xi_1(h, I, 5)$:

$$\nabla h(I) \neq 0\,;\quad v \neq 0\,;\quad h^1[v] = h^2[v,v] = h^3[v,v,v] = h^4[v,v,v,v] = 0\,, \tag{26.4.8}$$

so that $\Xi_1(h, I, 5) = \Psi_1(5)$ by Definition 4.
Now, since $\beta_2 = 5$ as it was in the case $n = 4$, we have exactly the same construction and we can write $\Xi_2(h, I, 5)$ in the same form:

$$\begin{cases} \nabla h(I) \neq 0\ ;\ \ h^1[u] = h^1[v] = \Pi_{\Lambda_2^I} h^2[v, \cdot] = \Pi_{\Lambda_2^I}\left(2\alpha h^2[u, \cdot] + h^3[v, v, \cdot]\right) = 0 \\ \Pi_{\Lambda_2^I}(6\beta h^2[u, \cdot] + 6\alpha h^3[u, v, \cdot] + h^4[v, v, v, \cdot]) = 0 \\ \Pi_{\Lambda_2}(24\gamma h^2[u, \cdot] + 24\beta h^3[v, u, \cdot] + 12\alpha^2 h^3[u, u, \cdot] \\ +12\alpha h^4[v, v, u, \cdot] + h^5[v, v, v, v, \cdot]) = 0\ . \end{cases} \tag{26.4.9}$$

with $u := A^2, v := A^1 + b_{21}A^2$ two linearly independent vectors and $\alpha := b_{22}, \beta := b_{23}, \gamma := b_{24}$ three real parameters. We now construct $\Xi_3(h, I, 5)$. As usual, we consider the Taylor polynomial $P_5^4(x_1, x_2, x_3)$ of the restriction of the function h to the subspace Λ_3^I up to order $\beta_3 = 4$. By calculating $\nabla P_5^4(x) = (\partial_{x_1} P_5^4(x), \partial_{x_2} P_5^4(x), \partial_{x_3} P_5^4(x))$ along the curve

$$x_1 = t\ ;\ \ x_2 = b_{21}t + b_{22}t^2 + b_{23}t^3\ ;\ \ x_3 = b_{31}t + b_{32}t^2 + b_{33}t^3$$

and by setting equal to zero all the coefficients of the resulting polynomial in t up to order $\beta_3 - 1 = 3$, one has that

1. The linear terms yield $\Pi_{\Lambda_3^I}(h^2[A^1 + b_{21}A^2 + b_{31}A^3, \cdot]) = 0$;
2. The quadratic terms yield

$$\begin{aligned} \Pi_{\Lambda_2^I}\,(\, & h^2[b_{22}A^2 + b_{32}A^3, \cdot] \\ & + h^3[A^1 + b_{21}A^2 + b_{31}A^3, A^1 + b_{21}A^2 + b_{31}A^3, \cdot]\,) = 0\ ; \end{aligned} \tag{26.4.10}$$

3. The cubic terms yield

$$\begin{aligned} & \Pi_{\Lambda_3^I}(6h^2[2b_{23}A^2 + 2b_{33}A^3, \cdot] + 6h^3[b_{22}A^2 + b_{23}A^3, A^1 + b_{21}A^2 + b_{31}A^3, \cdot] \\ & \quad +h^4[A^1 + b_{21}A^2 + b_{31}A^3, A^1 + b_{21}A^2 + b_{31}A^3, A^1 + b_{21}A^2 + b_{31}A^3, \cdot]) \\ & \quad\ = 0\ ; \end{aligned} \tag{26.4.11}$$

   where A_1, A_2, A_3 are a basis for Λ_3^I.

Therefore, $\Xi_3(h, I, 5)$ can be compactly formulated as

$$\begin{cases} \nabla h(I) \neq 0\ ;\ \ h^1[u] = h^1[v] = h^1[w] = 0\ ;\ \ \Pi_{\Lambda_3^I} h^2[v, \cdot] = 0 \\ \Pi_{\Lambda_3^I}(h^2[\alpha u + \beta w, \cdot] + h^3[v, v, \cdot]) = 0 \\ \Pi_{\Lambda_3^I}(6h^2[\gamma u + \delta w] + 6h^3[\alpha u + \beta w, v, \cdot] + h^4[v, v, v, \cdot]) = 0 \end{cases}\ . \tag{26.4.12}$$

with $u := A^2, v := A^1 + b_{21}A^2 + b_{31}A^3, w := A^3$ three linearly independent vectors and $\alpha := b_{22}, \beta := b_{32}, \gamma := b_{23}, \delta := b_{33}$ four real parameters.
Finally, we construct $\Xi_4(h, I, 5)$ in the same usual way.
As usual, we consider the Taylor polynomial $P_5^4(x_1, x_2, x_3)$ of the restriction of the function h to the subspace Λ_4^I up to order $\beta_4 = 2$.
By calculating $\nabla P_5^4(x) = (\partial_{x_1} P_5^4(x), \partial_{x_2} P_5^4(x), \partial_{x_3} P_5^4(x))$ along the curve

$$x_1(t) = t\ ;\quad x_2(t) = b_{21}t\ ;\quad x_3(t) = b_{31}t\ ;\quad x_4(t) = b_{41}t$$

and by setting equal to zero all the coefficients of the resulting polynomial in t up to order $\beta_4 - 1 = 1$, $\Xi_4(h, I, 5)$ reads

$$\nabla h(I) \neq 0\ ;\quad h^1[u] = h^1[v] = h^1[w] = h^1[x] = 0\ ;\quad \Pi_{\Lambda_4} h^2[v, \cdot] = 0\ , \qquad (26.4.13)$$

with $v = A^1 + b_{21}A^2 + b_{31}A^3 + b_{41}A^4, u = A^2, w = A^3, x = A^4$. With the help of definitions 4, 12 and 14, we see that $\sigma_5^5(I) = \Psi_1(5) \cup \Psi_2(5) \cup \Psi_3(5) \cup \Psi_4(5)$ so that, as a consequence of Lemma 5 and of Theorem 2, one has $\Sigma_5^5(I) = \bar{\sigma}_5^5(I) \subset \Psi_1^*(5) \cup \Psi_2^*(5) \cup \Psi_3^*(5) \cup \Psi_4^*(5)$. This, together with Theorem 2, implies Theorem 6. □

Chapter 27

Final remarks

27.1 The case $n \geq 6$

As the computations in the previous sections showed (see e.g. the case $n = 2$ or ref. [111]), Nekhoroshev's construction on affine linear subspaces of dimension $m = 1$ always yields a subsystem $\Xi_1(h, I, n)$ requiring β_1-degeneracy condition. In other words, take an arbitrary integer $r \geq 2$ and compute coefficient β_1 on a one-dimensional subspace; if there exists $v \neq 0$ such that

$$\nabla h(I) \neq 0 \; ; \;\; h^1[v] = 0 \; ; \;\; h^2[v, v] = 0 \; ; \;\; ... \; ; \;\; h^{\beta_1}[v, ..., v] = 0 \qquad (27.1.1)$$

is satisfied, then the r-jet of h belongs to $\sigma_n^r(I)$, since it fulfills membership requirements on subspaces of dimension $m = 1$. On the other hand, algebraic conditions for steepness on jets of order strictly greater than three make sense only at those points I where the function h is three-jet degenerate, since three-jet non-degeneracy automatically implies steepness. By looking at the explicit expression for β_m in Definition 3 and by taking expression (22.0.3) for the maximal index of steepness $\bar{\alpha}_m$ into account, one easily sees that $\beta_1 \leq 3$ for $r = 5$, $m = 1$ and $n \geq 6$. Therefore, the 5-jet of a function h with six or more degrees of freedom belongs to $\sigma_n^5(I)$ at those points I where h is three-jet degenerate. As a consequence, in this case Theorem 2 is helpless at establishing whether h is steep or not at those points where it is three-jet degenerate.

27.2 Genericity and further developments

As Nekhoroshev pointed out in refs. [94], [96] and as Theorem 2 shows, steepness is a generic property in the space of jets of a sufficiently high order r, since the codimension of the set containing the jets of all non-steep functions becomes positive for r sufficiently big. Such property is due to the fact that, for increasing r, one obtains more and more algebraic conditions that a function must satisfy in order to belong to such set. As

Nekhoroshev writes in ref. [94]: "Hamiltonians that fail to be steep at a non-critical point are infinitely singular: they satisfy an infinite number of conditions on their Taylor coefficients". This, in turn, is a straightforward consequence of Definition 3: when r increases, so does the order of the zero that the gradient of the tested function must possess on the minimal path γ so to stay in the bad set $\sigma_n^r(I)$. Indeed, since γ is a polynomial path, this implies that more and more coefficients of such polynomial must be set equal to zero, which yields an increasing number of algebraic conditions on the coefficients of the jet of the studied function.
In the present section, we give some examples of genericity for the sufficient conditions for steepness which we examined throughout the article.

Example 4. Quasi-convexity is a generic property in the space $\mathscr{P}(r,2)$ of polynomials of fixed degree $r \geq 2$ of two variables.

Proof. In case h is a non quasi-convex polynomial of order two in two variables, there exists $v \neq 0$ such that system

$$h^1[v] = 0\ ;\ \ h^2[v,v] = 0 \tag{27.2.1}$$

is satisfied. Moreover, v can be normalized to one since the system is homogeneous in such variable. Therefore, for all non quasi-convex functions h and for any integer $r \geq 2$, system (27.2.1) defines an algebraic set of codimension two in the cartesian space $\mathscr{P}(r,2) \times \mathbb{S}^1$ of polynomials and vectors. Since $\mathbb{S}^1$ has dimension one, by the Theorem of Tarski and Seidenberg, the projection of such algebraic set in the space of polynomials $\mathscr{P}(r,2)$ is semialgebraic and its codimension is no less than $2-1=1$. □

By following exactly the same startegy, one can prove also the two following

Example 5. Three-jet non-degeneracy is a generic property in the space $\mathscr{P}(r,3)$ of polynomials of fixed degree $r \geq 3$ of three variables.

Example 6. The sufficient conditions for steepness of Theorem 5 are generic in the space $\mathscr{P}(r,4)$ of polynomials of fixed degree $r \geq 4$ of four variables.

We remark that the minimal degree r of the polynomials for which genericity holds in the previous examples is the same one yielded by formula (4.1.2) in Theorem 2. Therefore, generic conditions for steepness for polynomials of arbitrary degree can only be inferred if one is able to write sufficient conditions for jets of any order. Such task is not straightforward and will be investigated in future works.

27.3 On the three-jet non-degeneracy condition

By closely looking at the algebraic form of the sets $\Psi_m^*(n)$ for $n \in \{2,3,4,5\}$ and $m \in \{1,...,n-1\}$, which was developed in the previous sections, one easily sees that any function whose jet belongs to any of these sets must be 3-jet degenerate. Therefore, if a function depending on a fixed number n of degrees of freedom is three-jet non degenerate, it belongs to the complementary of all sets $\Psi^*_m(n)$, with $m \in \{1, ..., n-1\}$. Since for fixed $n \in \{2, 3, 4, 5\}$ the bad set $\sigma^5_n(I)$ is contained in the union of closed sets $\cup_{m\in\{1,...,n-1\}} \Psi^*_m(n)$, by Theorem 2 one has that all three-jet non-degenerate functions depending on $n = 2, 3, 4, 5$ degrees of freedom are steep. We conjecture that for functions depending on $n \geq 6$ degrees of freedom the same result can be proved by closely looking at the algebraic form of the sets defining the bad set $\sigma^r_n(I)$, for a sufficiently high value of the order r. This would constitute an alternative strategy for proving the steepness of three-jet non-degenerate functions with respect to the one contained in [47].

Finally, by following a similar reasoning as in subsection 27.1, for $r = 3$ one obtains $\beta_1 \leq 3$ for $n \geq 2$, so that the set of jets of order three satisying the conditions for steepness of Theorem 2 is contained in the set of three-jet non degenerate jets. Therefore, three-jet non-degeneracy yields a wider set of steep functions with respect to the construction of Theorem 2.

Part VI

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Part VII

Appendices

Appendix A

Tools of real-algebraic geometry

The goal of this appendix is to provide the reader with an overview of some standard results of real-algebraic geometry that are used throughout the present work. The interested reader can find a complete exposition in [27] and [18].

A.1 Semi-algebraic sets and semi-algebraic functions

Definition A.1.1. A set $A \subset \mathbb{R}^n$ is said to be semi-algebraic if it can be written in the form

$$\bigcup_{i=1}^{s} \{ R_i(x) = 0, Q_{i1}(x) < 0, \dots, Q_{i\, r_i}(x) < 0 \} , \tag{A.1.1}$$

where $R_i, Q_{i1}, ..., Q_{i\, r_i} \in \mathbb{R}[x]$.

Remark A.1.1. If only equalities are present in (A.1.1), A is said to be algebraic.

It is clear that the polynomials generating a given semi-algebraic set A are not uniquely determined, nor is their number. However, one can introduce a unique quantity associated to a semi-algebraic set, namely

Definition A.1.2 (Diagram)**.** For any semi-algebraic set $A \subset \mathbb{R}^n$, we denote by

- $k_1(A)$ the minimal number of polynomials $R_i(X), Q_{ij}(X) \in \mathbb{R}[X]$ that are necessary in order to determine A as in (A.1.1).
- $k_2(A)$ the minimal value that the sum $\sum_{i=1}^{s} \deg(\widehat{R}_i) + \sum_{i=1}^{s} \sum_{j=1}^{r_i} \deg(\widehat{Q}_{ij})$ can attain, with $\widehat{R}_i(X), \widehat{Q}_{ij}(X) \in \mathbb{R}[X]$ determining A as in (A.1.1).

Following [36] (Def. 9.1), we call diagram of A the quantity

$$\operatorname{diag}(A) := k_1(A) + k_2(A) . \tag{A.1.2}$$

Semi-algebraic sets are stable under projection, namely

Theorem A.1.1. *(Tarski and Seidenberg, quantitative version)*

Take $n, m > 0$ and let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a semi-algebraic set. We indicate by $\Pi_n : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, (x, y) \longmapsto x$ the projector onto the first n coordinates. Then, the set $\Pi_n(A)$ is semi-algebraic and its diagram depends only on $\mathrm{diag}(A)$*, m and n.*

The classic versions of the Theorem of Tarski and Seidenberg do not usually make any reference to the diagram of the projected set. The statement given here can be found in [36] (Proposition 9.2) and its proof is contained in [17].

The Theorem of Tarski and Seidenberg is fundamental in order to demonstrate the following results (see [27] for proofs)

Proposition A.1.1. *The complementary of a semi-algebraic set $A \subset \mathbb{R}^n$ is semi-algebraic, and its diagram depends only on the diagram of A.*

Proposition A.1.2. *The closure, the interior and the boundary of a semi-algebraic set $A \subset \mathbb{R}^n$ are semi-algebraic and their diagrams depend only on the diagram of A.*

Proposition A.1.3. *Let A be a semi-algebraic set of $\mathbb{R}^n$. Then, indicating with $\overline{A}$ the closure of A, one has that* $\mathrm{diag}(\overline{A})$ *depends only on* $\mathrm{diag}(A)$*, and*

$$\dim A = \dim(\overline{A}) \ .$$

The notion of semi-algebraicness can be easily extended to functions by making reference to their graphs, namely

Definition A.1.3. Let $A \subset \mathbb{R}^n$ and $B \in \mathbb{R}^m$ be semi-algebraic sets. A map $\varphi : A \longrightarrow B$ is said to be semi-algebraic if $\mathrm{graph}(\varphi)$ is a semi-algebraic set of $\mathbb{R}^n \times \mathbb{R}^m$.

Semi-algebraic functions are piecewise algebraic, namely one has

Proposition A.1.4. *Let A be a semialgebraic subset $A \subset \mathbb{R}^n$ and $\varphi : A \longrightarrow \mathbb{R}$ be a semi-algebraic function of diagram $d > 0$. There exist a positive integer $M(d)$, a partition of A into a finite number of semi-algebraic sets $A_i, i = 1, \dots, m$, with $m \leq M(d)$, and for every value of i there exists a polynomial $S_i(X, Y)$ in $n + 1$ variables such that, for every x in A_i, $S_i(x, Y)$ is not identically zero and solves $S_i(x, \varphi(x)) = 0$.*

Proof. Except for the existence of the bound $M = M(d)$, the proof can be found in ref. [27] (Lemma 2.6.3). Assume, by absurd, that the bound $M(d)$ does not exist; then, one can find a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of semi-algebraic functions of A with diagram d such that, for each fixed $j \in \mathbb{N}$, the Proposition holds with the minimal number of pieces in the partition of A being equal to m_j, and $m_j \longrightarrow +\infty$. In particular, one can write $A = \sqcup_{i=1}^{m_j} A_{i,j}$ for any $j \in \mathbb{N}$, and there exist polynomials $S_{i,j}(X_1, \dots, X_n, Y)$ with the required properties. On the one hand, for any given $j \in \mathbb{N}$ and $i \in \{1, \dots, m_j\}$, by decomposition (A.1.1), one has that $\mathrm{graph}(\varphi_j|_{A_{i,j}})$ is the finite union of sets of the kind

$$\left\{(x, y) \in A_{i,j} \times \mathbb{R} \mid S_{i,j}(x, y) = 0, Q_{i1,j}(x, y) < 0, \dots, Q_{ir_i,j}(x, y) < 0\right\} \qquad \text{(A.1.3)}$$

for some polynomials $Q_{i1,j}, \ldots, Q_{ir_i,j} \in \mathbb{R}[X_1, \ldots, X_n, Y]$. On the other hand, one has the disjoint union

$$\mathrm{graph}(\varphi_j) = \bigsqcup_{i=1}^{m_j} \mathrm{graph}(\varphi_j|_{A_i,j}) , \tag{A.1.4}$$

which is a consequence of the fact that $A = \sqcup_{i=1}^{m_j} A_{i,j}$ is a partition.

By Def. A.1.2, formulas (A.1.3)-(A.1.4), and the fact that m_j is the minimal number of pieces in the partition of A for φ_j, one has $\mathrm{diag}(\varphi_j) \geq m_j$ and, for sufficiently high j, one has also $m_j > d$ since $m_j \longrightarrow +\infty$, in contradiction with the hypothesis $\mathrm{diag}(\varphi_j) = d$ for any $j \in \mathbb{N}$. This concludes the proof.

□

An immediate consequence of Proposition A.1.4 is the following

Corollary A.1.1. *Let $A \subset \mathbb{R}$ be an interval (finite or infinite) and $\varphi : A \longrightarrow \mathbb{R}$ be a semi-algebraic function of diagram $d > 0$. There exist a positive integer $M(d)$ and an interval $\mathcal{I}$ of length $|A|/M(d)$ over which the function φ is algebraic, namely there exists a polynomial $S(X,Y)$ in $n+1$ variables such that, for every x in $\mathcal{I}$, $S(x,Y)$ is not identically zero and solves $S(x,\varphi(x)) = 0$.*

Among semi-algebraic functions, an important class is that of Nash functions:

Definition A.1.4. Let A be an open semi-algebraic subset of $\mathbb{R}^n$. A semi-algebraic function $\varphi : A \longrightarrow \mathbb{R}$ belonging to the C^∞ class is said to be a Nash function. The set of Nash functions on A is a ring under the usual operations of sum and function multiplication.

Moreover, if we define analytic-algebraic functions as those real-analytic functions f defined on an open semi-algebraic set $A \subset \mathbb{R}^n$ and satisfying $P(x, f(x)) = 0$ for some polynomial P of $n+1$ variables and for all $x \in A$, it turns out that

Proposition A.1.5 (ref. [27], Prop. 8.1.8)**.** *A function $\varphi : A \longrightarrow \mathbb{R}$ is Nash on A if and only if it is analytic-algebraic.*

Another important property of more general complex analytic-algebraic functions is stated in the following

Proposition A.1.6. *Let $D \subset \mathbb{C}$ be an open, bounded domain. An analytic-algebraic function $f : D \longrightarrow \mathbb{C}$, whose graph solves a polynomial $S \in \mathbb{C}[z,w]$ of degree $k \in \mathbb{N}$, is k-valent: that is, if f is not constant then each value of* $\mathrm{Im}(f)$ *is the image of at most k points in D.*

Proof. Assume, by absurd, that f is non-constant and that there exists $w_0 \in \mathrm{Im}(f)$ which is the image of at least $p > k$ points in D. The polynomial $S^{w_0}(z) := S(z, w_0)$ would admit $p > k$ roots while $\deg(S^{w_0}) \leq k$ by hypothesis. The Fundamental Theorem of Algebra ensures that S^{w_0} must be identically zero and one has the factorization $S(z,w) = (w - w_0)^\alpha \widehat{S}(z,w)$, where $\alpha \in \{1, ..., k\}$, while $\widehat{S}$ cannot be divided by

$(w-w_0)$ in $\mathbb{C}[z,w]$. Since f is analytic and not constant, then the preimage $f^{-1}(\{w_0\})$ is a finite set and the graph of f must fulfill $\widehat{S}(z,f(z))=0$ out of $f^{-1}(\{w_0\})$. By continuity, one has $\widehat{S}(z,f(z))=0$ on the whole domain of definition of f since $f^{-1}(\{w_0\})$ is finite. But $\deg \widehat{S}^{w_0} \le k$, with $\widehat{S}^{w_0}(z) := \widehat{S}(z,w_0)$, and $\widehat{S}^{w_0}$ admits more than k roots, hence the previous argument ensures that $\widehat{S}$ can be divided by $(w-w_0)$, in contradiction with the construction. □

Semi-algebraicness is preserved by composition and inversion of semi-algebraic maps. In the following propositions, $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are supposed to be semi-algebraic sets.

Proposition A.1.7. *Let $\varphi : A \longrightarrow B$ a semi-algebraic map. If $S \subset A$ and $T \subset \varphi(A)$ are semi-algebraic, so are $\varphi(S)$ and the inverse image $\varphi^{-1}(T)$. Moreover, their diagrams depend only on the diagram of graph(φ).*

Proposition A.1.8. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two semi-algebraic functions. Then $f \circ g$ is semi-algebraic and the diagram of its graph depends only on the diagram of graph(f) and on the diagram of graph(g).*

Proposition A.1.9. *Let $f : A \longrightarrow B$ be an injective semi-algebraic function. Then, its inverse $f^{-1} : f(A) \longrightarrow A$ is semi-algebraic and the diagram of its graph depends only on the diagram of graph(f).*

Proposition A.1.10. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \longrightarrow \mathbb{R}$ be a semi-algebraic function differentiable in I. Then its derivative f' is a semi-algebraic function and its diagram only depends on the diagram of graph(f).*

We refer to [27] for the proofs of these statements. The dependence of the diagrams on the diagram of the initial function is, once again, a consequence, of the quantitative version A.1.1 of the Theorem of Tarski and Seidenberg.

Finally, we give the following statement, which will prove to be helpful in our work

Proposition A.1.11. *(see e.g. ref. [74], pag. 23-24) Let $f : A \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}^m$ be semi-algebraic functions and suppose that f is bounded from below. Then*

$$\varphi : g(A) \longrightarrow \mathbb{R} \qquad y \longmapsto \inf_{x \in g^{-1}(y)} f(x)$$

is semi-algebraic and the diagram of its graph depends only on the diagrams of graph(f) and graph(g).

A.2 Analytic reparametrization of semi-algebraic sets

Generally speaking, the reparametrization of a semi-algebraic set A is a subdivision of A into semi-algebraic pieces A_j each of which is the image of a semi-algebraic function of the unit cube. On the one hand, it is possible to cover the whole of A if one asks for the covering functions to be of finite regularity, with a uniform control on their derivatives (see [67]). On the other hand, if one requires analyticity of the covering functions together with a uniform control on their derivatives, it is only possible to cover A up to a "small" subset.

Hereafter, we state this result only in the case we need, that is for reparametrizations of graphs of algebraic functions, referring to [116] for the general theory.

It is known that algebraic functions can only have two type of complex singularities: ramification points and poles (where the function may also ramify). If we denote by d the diagram of an algebraic function, the number of its complex singularities is bounded by a quantity depending only on d (see e.g. [16]). It is exactly the neighborhoods of these singularities that cannot be analytically covered.

Definition A.2.1. Let $\delta > 0$ and $g : I := [-1, 1] \longrightarrow \mathbb{R}$ be an algebraic function. An analytic δ-reparametrization of g consists of

- A finite number N of open subintervals U_i of I, with $\text{length}(U_i) \leq 2\delta$ for any $i = 1, ..., N$.
- A partition of $I \setminus \cup_{i=1}^{N} U_i$ into a finite number of subsegments Δ_j, $j = 1, ..., M$, together with a collection of analytic maps $\psi_j : I \longrightarrow \Delta_j$ such that for any $j \in \{1, ..., N\}$

  1. ψ_j is an affine reparametrization of the segment Δ_j.
  2. ψ_j and $g \circ \psi_j$ are both holomorphic in $\mathcal{D}_3 := \{z \in \mathbb{C} : |z| \leq 3\}$.
  3. ψ_j and $g \circ \psi_j$ both satisfy a Bernstein inequality, that is

$$\max_{z \in \mathcal{D}_3} |\psi_j(z) - \psi_j(0)| \leq 1 \qquad \max_{z \in \mathcal{D}_3} |g \circ \psi_j(z) - g \circ \psi_j(0)| \leq 1 \, . \quad \text{(A.2.1)}$$

Theorem A.2.1. *(Yomdin, [116] Th. 3.2) Let d be a positive integer. There exist constants $\Upsilon_1 = \Upsilon_1(d)$ and $\Upsilon_2 = \Upsilon_2(d)$ such that for each algebraic function $g(x)$ of diagram d defined on I satisfying $0 \leq g(x) \leq 1$ and for each $\delta > 0$ there is an analytic δ-reparametrization of g with the number N of the removed intervals bounded by Υ_1 and the number M of the covering maps bounded by $\Upsilon_2 \log_2(1/\delta)$. Each of the removed intervals is centered at the real part of a complex singularity of $g(x)$.*

Moreover, one has the following auxiliary result concerning the distance of each interval Δ_j to the singularities of the complex extension of the function g

Proposition A.2.1. *(Yomdin, [116], Lemma 3.6) The center of any of the segments Δ_j in the partition given by Theorem A.2.1 is at distance no less than $3 \times |\Delta_j|$ from any complex singularity of g.*

Appendix B

Quantitative local inversion theorem

We start by stating a Lipschitz inverse function Theorem. Its proof can be found, for example, in [65] (Th. 14.6.6).

Theorem B.0.1. *Let U be an open subset of a Banach space E and that $k : U \longrightarrow E$ is a Lipschitz mapping with constant $K < 1$. Set $h(x) = x + k(x)$. If the closed ball $\overline{\mathtt{B}}_\varepsilon(x)$ of radius ε around x is contained in U then $\overline{\mathtt{B}}_{(1-K)\varepsilon}(h(x)) \subset h(\overline{\mathtt{B}}_\varepsilon(x)) \subset \mathtt{B}_{(1+K)\varepsilon}(h(x))$. The mapping h is a homeomorphism of U onto $h(U)$, h^{-1} is a Lipschitz mapping with constant $1/(1-K)$, and $h(U)$ is an open subset of E.*

This result is crucial in order to prove an analytic inverse function theorem, namely

Theorem B.0.2. *Take a function $f \in C^\omega(\overline{\mathcal{D}}_R(0))$ and a point $z^* \in \mathcal{D}_{R/2}(0)$ satisfying $f'(z^*) \neq 0$. Then, f is invertible in the closed disk $\overline{\mathcal{D}}_{R'/16}(z^*)$ and its inverse f^{-1} is analytic in $\overline{\mathcal{D}}_{|f'(z^*)|R'/8}(f(z^*))$, where*

$$R' := \frac{1}{2} \times \min \left\{ R, \frac{|f'(z^*)|}{\max_{\overline{\mathcal{D}}_R(0)} |f''|}, \right\} .$$

Proof. We define $h(z) := \dfrac{f(z) - f(z^*)}{f'(z^*)} + z^*$, $k(z) := h(z) - z$; both these functions are obviously holomorphic in $\overline{\mathcal{D}}_{R/2}(z^*)$. Since $k'(z) = h'(z) - 1 = \dfrac{f'(z)}{f'(z^*)} - 1 = \dfrac{f'(z) - f'(z^*)}{f'(z^*)}$, one has $|k'(z)| = \dfrac{|f'(z) - f'(z^*)|}{|f'(z^*)|} \leq \max_{\overline{\mathcal{D}}_R(0)} |f''| \dfrac{|z - z^*|}{|f'(z^*)|}$. If we choose to consider only the $z \in \mathcal{D}_{R'}(z^*)$, we obtain that k is $\dfrac{1}{2}$–Lipschitz on this set.

At this point, we exploit Theorem B.0.1 and we have that the function $h(z) = k(z)+z$ is a homeomorphism of $\mathcal{D}_{R'}(z^*)$ onto its image. Moreover, one has $\overline{\mathcal{D}}_{R'/8}(h(z^*)) \subset$

$h(\overline{\mathcal{D}}_{R'/4}(z^*)) \subset \overline{\mathcal{D}}_{3R'/8}(h(z^*))$ and, since $h(z^*) = z^*$, this yields

$$\overline{\mathcal{D}}_{R'/8}(z^*) \subset h(\overline{\mathcal{D}}_{R'/4}(z^*)) \subset \overline{\mathcal{D}}_{3R'/8}(z^*) \,. \tag{B.0.1}$$

We can define f^{-1} by exploiting h and its inverse, namely

$$z = h^{-1}(h(z)) = h^{-1}\left(z^* + \frac{f(z) - f(z^*)}{f'(z^*)}\right) =: f^{-1}(f(z)) \tag{B.0.2}$$

Indeed, by expressions (B.0.1) and (B.0.2), we see that, if we choose

$$\left|z^* + \frac{f(z) - f(z^*)}{f'(z^*)} - z^*\right| \le \frac{R'}{8} \,,$$

that is $|f(z) - f(z^*)| \le |f'(z^*)|\dfrac{R'}{8}$, we have defined the inverse over the closed disc $\overline{\mathcal{D}}_{|f'(z^*)|R'/8}(f(z^*))$. Finally, we prove that

$$f(\overline{\mathcal{D}}_{R'/16}(z^*)) \subset \overline{\mathcal{D}}_{|f'(z^*)|R'/8}(f(z^*)) \,.$$

In order to see this, for $z \in \overline{\mathcal{D}}_{R'/16}(z^*)$ we consider the identity

$$f(z) - f(z^*) = f'(z^*)(z - z^*) + \int_0^1 f''(tz + (1-t)z^*)\,(z - z^*)^2\,(1-t)dt$$

that yields the estimate

$$\begin{aligned} |f(z) - f(z^*)| \le& |f'(z^*)||z - z^*| + \max_{\overline{\mathcal{D}}_R(0)} |f''|\,|z - z^*|^2 \\ \le& |f'(z^*)|\frac{R'}{16} + \max_{\overline{\mathcal{D}}_R(0)} |f''|\frac{R'^2}{256} \le |f'(z^*)|\frac{R'}{8} \,, \end{aligned} \tag{B.0.3}$$

where the last estimate is a consequence of the definition of R'.

The fact that f^{-1} inherits the same regularity of f is a standard consequence of the classic local inversion theorem. □

Appendix C

Auxiliary Lemmas for the genericity of steepness

C.1 Two elementary properties of Lie Groups

We refer to [81] (Cor. 21.6, Th. 21.10) for proofs.

Proposition C.1.1. *Every continuous action by a compact Lie group on a manifold is proper.*

Theorem C.1.1 (Quotient manifold)**.** *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold $\mathcal{M}$. Then the orbit space $\mathcal{M}/G$ is a topological manifold of dimension equal to* $\dim \mathcal{M} - \dim G$ *and has a unique smooth structure with the property that the quotient map $\pi : \mathcal{M} \longrightarrow \mathcal{M}/G$ is a smooth submersion.*

C.2 Three auxiliary Lemmas

The following Lemma is an elementary criterion to establish when the projection of a closed set is still closed.

Lemma C.2.1. *Let E be a metric space, K a compact subset of some metric space and Δ a closed subset of $E \times K$. Then, the projection of Δ on E, indicated by $\Pi_E(\Delta)$, is closed.*

Proof. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in $\Pi_E(\Delta)$ converging to a point $\bar{p}$ and $\{k_n\}_{n\in\mathbb{N}}$ a sequence in K satisying $(p_n, k_n) \in \Delta$. Since K is a compact subset of some metric space, one can extract a subsequence $\{k_{n_l}\}_{l\in\mathbb{N}}$ converging to a point $\bar{k} \in K$. Hence, the sequence $\{(p_{n_l}, k_{n_l})\}_{l\in\mathbb{N}}$ in Δ converges to $(\bar{p}, \bar{k}) \in \Delta$, since Δ is closed. This implies that $\bar{p}$ belongs to $\Pi_E(\Delta)$, which is therefore closed. □

The following statement is a known Theorem due to Bézout (see [79], Th. 3.4a).

Lemma C.2.2. *For any couple of positive integers k_1, k_2 consider two non-zero irreducible, non-proportional polynomials $Q_1 \in \mathcal{P}(k_1)$ and $Q_2 \in \mathcal{P}(k_2)$. Then the system $Q_1(z,w) = Q_2(z,w) = 0$ has at most $k_1 \times k_2$ solutions.*

For the sake of completeness, we also state the following simple result on the codimension of the zero set of a non-null polynomial.

Lemma C.2.3. *Let n be a positive integer. Consider a non-null real polynomial $P \in \mathbb{R}[x]$ of the variables $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The zero set $Z_P := \{x \in \mathbb{R}^n | P(x) = 0\}$ is contained in a submanifold of codimension one in $\mathbb{R}^n$.*

Proof. If P is a non-zero constant, then there is nothing to prove.

If P is non-constant, the proof is by induction on the degree of P.

If $\deg P = 1$, then Z_P is a hyperplane, which is obviously a submanifold of codimension one.

Suppose, now, that the statement is true for polynomials of degree $k - 1 \geq 1$. Consider a polynomial of degree k, together with its associated open set of non-critical points $S_P := \{x \in \mathbb{R}^n \mid \nabla P(x) \neq 0\}$. On the one hand, locally around any point of $Z_P \cap S_P$ one can apply the implicit function theorem, so that $Z_P \cap S_P$ is indeed a submanifold of codimension one in $\mathbb{R}^n$. On the other hand, $\mathbb{R}^n \backslash S_P$ is the common zero set of the n polynomials $\partial P/\partial x_1, \dots, \partial P/\partial x_n$; moreover, since $\deg P = k \geq 2$, at least one among $\partial P/\partial x_1, \dots, \partial P/\partial x_n$ has degree $k - 1 \geq 1$. Hence, by hypothesis, $\mathbb{R}^n \backslash S_P$ is contained in a submanifold of codimension one in $\mathbb{R}^n$.

This proves that the set $Z_P \cap (\mathbb{R}^n \backslash S_P) \subset (\mathbb{R}^n \backslash S_P)$ is contained in a submanifold of codimension one in $\mathbb{R}^n$. Obviously, the thesis follows by the fact that

$$Z_P = (Z_P \cap (\mathbb{R}^n \backslash S_P)) \cup (Z_P \cap S_P) \, .$$

□

Appendix D

A Lemma on Riemann branches

The goal of this appendix consists in proving Lemma 13.0.1. We start by stating two standard results of algebraic geometry.

Lemma D.0.1. *For any pair of positive integers k_1, k_2 consider two non-zero irreducible, non-proportional polynomials $Q_1 \in \mathcal{P}(k_1)$ and $Q_2 \in \mathcal{P}(k_2)$. Then the system $Q_1(z,w) = Q_2(z,w) = 0$ has at most $k_1 \times k_2$ solutions.*

Lemma D.0.2. *For $k \geq 2$, let $Q(z,w) \in \mathcal{P}(r,n)$ be an irreducible polynomial. Then*

$$card\,\{z \in \mathbb{C} \mid \exists w \in \mathbb{C} : Q(z,w) = \partial_w Q(z,w) = 0\} \leq k\ .$$

The first Lemma is a simple corollary of Bézout's Theorem (see e.g. [79], Th. 3.4a), while the second Lemma is also known (see e.g. Proposition 1 and its proof in [92]).

With these tools, we can now give the proof of Lemma 13.0.1.

Proof. The lemma is trivial if S depends only on w since we have $\mathcal{N}_S = \emptyset$ in this case because $\mathtt{R}_S$ is composed of a finite number of Riemann branches which are horizontal lines over the z-axis.

If $S \in \mathcal{P}(r,n)$ depends only on z, then $\mathtt{R}_S = \{(z,w) \in \mathbb{C}^2 : z = z_0,\ \text{with } S(z_0) = 0\}$ and the thesis holds true since there are only vertical lines at the distinct roots of S (whose number is bounded by k) and no Riemann branches.

Let's now examine the case in which S depends on both variables where, up to multiplication by constant factors, any polynomial $S \in \mathcal{P}(r,n)$ can be factorized uniquely as

$$S(z,w) = q(z)\,\Pi_{i=1}^{m}(S_i(z,w))^{j_i} \tag{D.0.1}$$

for some $1 \leq j_i \leq k$, $1 \leq m \leq k$ and the S_i are non-constant, irreducible, mutually non-proportional polynomials.

We denote

$$\overline{S}(z,w) = \Pi_{i=1}^{m}(S_i(z,w))^{j_i} \text{ and } \widetilde{S}(z,w) = \Pi_{i=1}^{m} S_i(z,w) \tag{D.0.2}$$

and $\overline{S}^z(w) := \overline{S}(z,w)$, $\widetilde{S}^z(w) := \widetilde{S}(z,w)$ hence $\widetilde{S}^z \in \mathbb{C}[z][w]$ - with $\deg(\widetilde{S}^z) = \ell$ $\ell \in \{1,\ldots,k\}$ - and $a_\ell(z)$ is the corresponding leading coefficient.

We notice that decomposition D.0.1 and definition D.0.2 ensure that $\mathtt{R}_S$ is the union of the vertical lines $z = z^*$ with $q(z^*) = 0$ and of the Riemann surface $\mathtt{R}_{\overline{S}}$, moreover the Riemann surfaces $\mathtt{R}_{\overline{S}}$ and $\mathtt{R}_{\widetilde{S}}$ are identical.

Definition D.0.1 (Excluded points)**.** Taking decomposition D.0.1 into account, we define $\mathcal{N}_S \subset \mathbb{C}$ as the set of those points $z_0 \in \mathbb{C}$ that satisfy at least one of the following conditions

1. $$q(z_0) = 0 \qquad \text{(Vertical lines)}$$

2. There exists $w_0 \in \mathbb{C}$ such that for some $i \in \{1,...,m\}$
$$\begin{cases} S_i(z_0,w_0) = 0 \\ \partial_w S_i(z_0,w_0) = 0 \end{cases} \qquad \text{(Ramification points)}$$

3. There exists $w_0 \in \mathbb{C}$ such that for some $i,j \in \{1,...,m\}, i \neq j$
$$\begin{cases} S_i(z_0,w_0) = 0 \\ S_j(z_0,w_0) = 0 \end{cases} \qquad \text{(Intersection of graphs)}$$

4. $$a_\ell(z_0) = 0 \qquad \text{(Poles)}$$

Henceforth, we prove that over $\mathbb{C}\backslash\mathcal{N}_S$ the conclusions of Lemma 13.0.1 are valid and that we can choose the set $\mathcal{N}_S$ as the excluded points for S.

To see this, we fix a point $z^* \in \mathbb{C}\backslash\mathcal{N}_S$.

By negation of condition (1), we have $q(z) \neq 0$ in the vicinity of z^*, hence the vertical lines are excluded from the algebraic curve $\mathtt{R}_S$ in the vicinity of z^*.

Then, we also notice that for any value w^* such that $\overline{S}^{z^*}(w^*) = 0$, by decomposition (D.0.1) and negation of condition (3), one must have $S_i(z^*,w^*) = 0$, for exactly one $i \in \{1,...,m\}$. Hence, by negation of condition (2) at (z^*,w^*), we can apply the implicit function theorem and there exists an open neighbourhood V around (z^*,w^*) such that $\mathtt{R}_S \cap V = \mathtt{R}_{\overline{S}} \cap V = \mathtt{R}_{\widetilde{S}} \cap V = \mathtt{R}_{S_i} \cap V$ is the graph of an unique holomorphic function.

Finally, the negation of condition (4) in a neighbourhood of z^* ensures that the polynomial $\widetilde{S}^z$ admits $\ell \leq k$ complex roots counted with multiplicity for z in the vicinity of z^*. With $z^* \in \mathbb{C}\backslash\mathcal{N}_S$, a direct computation ensures that the discriminant of $\widetilde{S}^{z^*}$ is non-zero since $\widetilde{S}^{z^*}$ and its derivative cannot have common roots, hence $\widetilde{S}^z$ admits simple roots for z in the vicinity of z^*.

This implies the existence of a neighborhood V of $z^* \in \mathbb{C}\backslash\mathcal{N}_S$ such that the algebraic curve $\mathtt{R}_S \cap V \times \mathbb{C} = \mathtt{R}_{\widetilde{S}} \cap V \times \mathbb{C}$ is the union of exactly $\ell \leq k$ disjoint graphs of holomorphic branches.

Hence, over a simply connected domain $\mathtt{D} \subset \mathbb{C} \backslash \mathcal{N}_S$, branch cuts can be avoided and the Riemann surface $\mathtt{R}_S$ is the finite union of at most k disjoint graphs of holomorphic functions.

Conversely, consider a simply connected complex domain $\mathtt{D}$ such that $\mathtt{R}_S \cap \mathtt{D} \times \mathbb{C}$ is the finite union of $\ell \in \{1, \ldots, k\}$ disjoint graphs of functions $h_1(z), \ldots, h_\ell(z)$ holomorphic over $\mathtt{D}$.

For a fixed point $z^* \in \mathtt{D}$, the polynomial S^{z^*} admits ℓ roots, and decomposition D.0.1 ensures that we have $q(z^*) \neq 0$. Moreover, the discriminant of $\widetilde{S}^z$ might be zero only at a finite number of points (since the discriminant is itself a polynomial) but we always have ℓ roots for $\widetilde{S}^z$ with $z \in \mathtt{D}$ as a consequence of our assumption that the Riemann leaves are distinct. Hence, the roots are simple and the degree of $\widetilde{S}^z$ is constant equal to ℓ for all $z \in \mathtt{D}$. Consequently, the discriminant of $\widetilde{S}^z$ is non-zero and $a_\ell(z) \neq 0$ for all $z \in \mathtt{D}$.

Moreover, for any $i, j \in \{1, ..., m\}, i \neq j$ and for any $w \in \mathbb{C}$, we have either $S_i(z^*, w) \neq 0$ or $S_j(z^*, w) \neq 0$ otherwise two distincts Riemann leaves associated respectively to S_i and S_j would intersect.

Finally, we have the decomposition $\widetilde{S}(z, w) = a_\ell(z)(w - h_1(z)) \ldots (w - h_\ell(z))$ for $(z, w) \in \mathtt{D} \times \mathbb{C}$, and for $z \in \mathtt{D}$ we can check that S^z and its derivative cannot have common roots under our assumptions. Hence, for any $w \in \mathbb{C}$ and any $i \in \{1, \ldots, m\}$, we have either $S_i(z^*, w) \neq 0$ or $\partial_w S_i(z^*, w) \neq 0$.

Then, we prove that the cardinality of $\mathcal{N}_S$ is bounded by a quantity depending only on k.

Conditions (1) and (4) are polynomial equations of degree less than or equal to k (q factorizes all the terms in z and $a_\ell(z)$ is the coefficient of the term of highest degree in w), hence they have at most k solutions. By Lemma D.0.1, condition (2) is satisfied at most at k points. Since the index i in (2) can assume at most k values, this condition yields k^2 singularities. In the same way, Lemma D.0.2 says that condition (3) is satisfied at most at k^2 points. Since the indices i, j in condition (3) can each take at most k values and $i \neq j$, we get $k^2\binom{k}{2}$ solutions. The sum of the previous estimates yields a bound depending only on k. □

Appendix E

Tools of complex analysis

$\mathcal{O}(\mathcal{D})$ denotes the set of holomorphic functions over an open domain $\mathcal{D} \subset \mathbb{C}$.

Definition E.0.1 (Normal families)**.** A family $\mathcal{F} \subset \mathcal{O}(\mathcal{D})$ is said to be normal iff it is precompact, that is, if from any sequence in $\mathcal{F}$, one can extract a subsequence converging uniformly on the compact subsets of $\mathcal{D}$ to a function in $\mathcal{O}(\mathcal{D})$.

Definition E.0.2 (Locally bounded families)**.** A family $\mathcal{F} \subset \mathcal{O}(\mathcal{D})$ is said to be locally bounded if for any compact set $\mathcal{K} \subset \mathcal{D}$, there exists a constant $C_{\mathcal{K}} \geq 0$ such that $\max_{\mathcal{K}} |f| \leq C_{\mathcal{K}}$ for all $f \in \mathcal{F}$.

A criterion for establishing if a given family $\mathcal{F}$ is normal is Montel's theorem, which is a version of the Ascoli-Arzelà theorem for sets of holomorphic functions.

Theorem E.0.1 (Montel, see for example [112], Prop. 21.3)**.**
A family $\mathcal{F} \subset \mathcal{O}(\mathcal{D})$ is locally bounded if and only if it is normal.

From this theorem comes an important

Corollary E.0.1. *If $\{f_n\}_{n\in\mathbb{N}}$ is a locally bounded sequence of holomorphic functions over $\mathcal{D}$, then one can extract a subsequence that converges uniformly to a holomorphic function f on all the compact subsets of $\mathcal{D}$.*

Hereafter, we state some further classic results of complex analysis.

Theorem E.0.2 (Laurent series, [112] p. 130)**.**
Take two numbers $0 < r_1 < r_2$, $z_0 \in \mathbb{C}$ and $f \in \mathcal{O}(\mathtt{A}_{r_1,r_2}(z_0))$ where $\mathtt{A}_{r_1,r_2}$ is the annulus of radii r_1 and r_2 around z_0, then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \ ,$$

where the convergence is uniform over the compact subsets of $\mathtt{A}_{r_1,r_2}(z_0)$.

Corollary E.0.2 (Laurent classification of singularities, [112])**.**
Let $z_0 \in \mathbb{C}$ be a singularity of a holomorphic function f and consider the Laurent series $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k$, then

- *$z = z_0$ is a removable singularity iff $a_k = 0$ for any $k \leq -1$.*
- *given $m \in \mathbb{N}$, $z = z_0$ is a pole of order m iff $a_{-m} \neq 0$ and $a_k = 0$ for any $k \leq -m - 1$.*
- *$z = z_0$ is an essential singularity iff $a_k \neq 0$ for infinitely many negative integers k.*

Theorem E.0.3 (Casorati-Weierstrass Theorem, [112] p.127)**.**
Let $z_0 \in \mathbb{C}$ be an isolated essential singularity of a function $f \in \mathcal{O}(\Omega \setminus \{z_0\})$, then for every neighborhood V of z_0 in Ω, the image of $V \setminus \{z_0\}$ under f is dense in $\mathbb{C}$.

Theorem E.0.4 (Riemann's Theorem on removable singularities, [112] p.126)**.**
Take $r > 0$, $z_0 \in \mathbb{C}$ and $f \in \mathcal{O}(\dot{\mathcal{D}}_r(z_0))$ with f bounded.
Then

- $\lim_{z \longrightarrow z_0} f(z)$ *exists and is finite;*
- *the function $\tilde{f} : \mathcal{D}_r(z_0) \longrightarrow \mathbb{C}$,*

$$\tilde{f} := \begin{cases} f(z) & \text{if } z \in \dot{\mathcal{D}}_r(z_0) \\ \lim_{z \longrightarrow z_0} f(z) & \text{if } z = z_0 \end{cases}$$

  is holomorphic.

Theorem E.0.5 (Hurwitz, [112] p. 216)**.** *Suppose that $\mathcal{D}$ is a connected open set and that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of nowhere vanishing holomorphic functions over $\mathcal{D}$. If the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on compact subsets of $\mathcal{D}$ to a (necessarily holomorphic) limit function f, then either f is nowhere-vanishing over $\mathcal{D}$ or f is identically null.*

Appendix F

Auxiliary results on algebraic functions

F.0.1 On the dependence of the roots of a polynomial on its coefficients

It is a standard fact in the study of algebra that the roots of a monic complex polynomial of one variable depend continuously on its coefficients. The same result holds true for non-monic polynomials once one takes solutions at infinity into account by means of the compact identification of $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere. Without entering into too many details, we state the following result, whose proof can be found in [53].

Theorem F.0.1. *Let $P(w) = a_n w^n + a_{n-1} w^{n-1} + ... + a_0$ be a non-zero complex polynomial of degree $k \leq n$. Let $\xi_1, ..., \xi_r$ be its roots in $\mathbb{C}$ with $m_1, ..., m_r$ their respective multiplicities. Fix $\varepsilon > 0$ small enough and denote by $\mathcal{D}_\varepsilon(\xi_1), ..., \mathcal{D}_\varepsilon(\xi_r)$ the disjoint disks of radius ε centered at $\xi_1, ..., \xi_r$, such that $\mathcal{D}_\varepsilon(\xi_j) \subset \mathcal{D}_{1/\varepsilon}(0)$ for all $j \in \{1, ..., r\}$. Then, there exists $\delta(\varepsilon) > 0$ such that every complex polynomial $Q(w) = b_n w^n + b_{n-1} w^{n-1} + ... + b_0$ satisfying $|b_j - a_j| < \delta(\varepsilon)$ for all $j \in \{0, ..., n\}$ has m_i roots (counted with multiplicity) in each $\mathcal{D}_\varepsilon(\xi_i)$ for $i \in \{1, ..., r\}$ and* $\deg(Q) - k$ *roots belonging to the complement of $\mathcal{D}_{1/\varepsilon}(0)$.*

This theorem formalizes the intuitive idea that, if one takes a polynomial

$$Q(w) = a_n w^n + a_{n-1}, w^{n-1} + ... + a_0, \text{ with } a_n \neq 0$$

and makes $a_{k+1}, a_{k+2}, ..., a_n$ tend to zero while $a_k \neq 0$, then $n-k$ solutions "continuously go to infinity" and k solutions, counted with their multiplicities, "stay finite."

F.0.2 Application to sequences of algebraic functions

Lemma F.0.1. *Take an open bounded set $\mathbb{U} \subset \mathbb{C}$, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of algebraic functions in $\mathcal{O}(\mathbb{U})$ associated to polynomials of degree $k \in \mathbb{N}$ and converging in $\mathbb{U}$ to a* *function $g \in \mathcal{O}(\mathbb{U})$. Then g is an algebraic function. Moreover, there exists a sequence of polynomials $\{Q_n \in \mathcal{P}(r,n)\}_{n\in\mathbb{N}}$ solving the graphs of the functions in $\{g_n\}_{n\in\mathbb{N}}$ which converges to a polynomial $Q \in \mathbb{C}[z,w]$ that solves graph(g) everywhere in $\mathbb{U}$.*

Proof. For any $n \in \mathbb{N}$, the graph of the function g_n satisfies $S_n(z, g_n(z)) = 0$ for some polynomial $S_n \in \mathcal{P}(k)\backslash\{0\}$ and the equation $S_n(z, g_n(z)) = 0$ is invariant when S_n is multiplied by any non-zero constant. Without loss of generality, one can choose an arbitrary norm $||\cdot||$ in $\mathcal{P}(r,n) \simeq \mathbb{C}^m$, with $m = (k+1)(k+2)/2$, and consider the sequence of polynomials $\{Q_n\}_{n\in\mathbb{N}}$ associated to $\{g_n\}_{n\in\mathbb{N}}$ by defining, for any $n \in \mathbb{N}$:

$$Q_n(z,w) := \frac{S_n(z,w)}{||S_n||} \text{ with } Q_n(z, g_n(z)) = 0 \text{ and } Q_n \in \mathbb{S}^m \tag{F.0.1}$$

where $\mathbb{S}^m$ denotes the unitary sphere in $\mathcal{P}(r,n) \simeq \mathbb{C}^m$. By compactness of $\mathbb{S}^m$, there exists a subsequence $\{Q_{n_j}\}_{j\in\mathbb{N}}$ converging to a polynomial $Q \in \mathbb{S}^m$. Moreover, if we denote $Q^z_{n_j}(w) := Q_{n_j}(z,w)$ and $Q^z(w) := Q(z,w)$ - hence $Q^z_{n_j}$ and Q^z belong to $\mathcal{Q}(k)$ for any fixed $z \in \mathbb{C}$ - we have the convergence

$$\lim_{j\longrightarrow+\infty} ||Q^{z^*}_{n_j} - Q^{z^*}|| = 0 \tag{F.0.2}$$

for any fixed $z^* \in \mathbb{U}$ and with respect to any norm in $\mathcal{Q}(k)$.

The convergence in (F.0.2) and Theorem F.0.1 imply that the sequence $\{g_{n_j}(z^*)\}_{j\in\mathbb{N}}$ approaches a root of Q^{z^*} for any $z^* \in \mathbb{U}$. Since g_{n_j} converges over $\mathbb{U}$ to g, then $g(z)$ is a solution of $Q^z(w) = 0$ for any $z \in \mathbb{U}$.

Finally, since $g \in \mathcal{O}(\mathbb{U})$, then g is a Riemann branch of $Q \in \mathcal{P}(k)$ over $\mathbb{U}$. □

F.0.3 Non-existence of essential singularities for algebraic functions

Proposition F.0.1. *An algebraic function f cannot have any essential singularity.*

Proof. By Lemma 13.0.1 and decomposition D.0.1, the singularities of f are included in the set $\mathcal{N}_{\overline{S}}$ of excluded points (see D.0.1). Hence, suppose by contradiction that $z_0 \in \mathcal{N}_{\overline{S}}$ is an essential singularity. Since the cardinality of $\mathcal{N}_{\overline{S}}$ is finite, z_0 is isolated. Then, the Casorati-Weierstrass Theorem (see Th. E.0.3) holds and, for any fixed $w_0 \in \mathbb{C}$, one can build a sequence $\{z_k\}_{k\in\mathbb{N}}$ converging to z_0 and such that

$$\lim_{k\longrightarrow+\infty} f(z_k) = w_0\ .$$

But w_0 is also a root of the one-variable polynomial $\overline{S}^{z_0}(w) := \overline{S}(z_0, w)$ since $f(z)$ is a Riemann branch of $\overline{S}$ in a punctured neighborhood centered at z_0 and

$$\overline{S}^{z_0}(w_0) := \overline{S}(z_0, w_0) = \lim_{k\longrightarrow+\infty} \overline{S}(z_k, f(z_k)) = 0.$$

This construction holds for any $w_0 \in \mathbb{C}$ and the polynomial $\overline{S}^{z_0}$ is null but, necessarily, $(z - z_0)$ is a factor of $\overline{S}$ and this is not possible with decomposition D.0.1. Hence, $f(z)$ cannot have an essential singularity at z_0. □

Appendix G

Smoothing estimates

Lemma G.0.1. *The derivatives of K satisfy*

$$\forall p \in \mathbb{N}, \quad \exists C_p \ : \left|\partial^\beta K(x)\right| \leq C_p \frac{e^{|\operatorname{Im} x|}}{(1+|x|_2)^p}, \ \forall |\beta| \leq p.$$

For the proof see [45, Lemma 9].

Lemma G.0.2. *Let $f \in C_b^\ell(\mathbb{A}^n)$, with $\ell \geq 1$, and let $\sum_{k\in\mathbb{Z}^n} \hat{f}_k(I)e^{ik\cdot\theta}$ be its Fourier series. Then, for any fixed $k \in \mathbb{Z}^n \backslash \{0\}$, there exists a uniform constant $C_{\mathrm{F}}(n,\ell)$ satisfying*

$$\left\|\hat{f}_k\right\|_{C^0(\mathbb{R}^n)} \leq C_{\mathrm{F}}(n,\ell)\frac{\|f\|_{C^q(\mathbb{A}^n)}}{|k|^q} \ , \tag{G.0.1}$$

where $q := \lfloor \ell \rfloor$.

Proof. Fix a multi-index $j = (j_1, ..., j_n) \in \mathbb{N}^n$ such that $|j|_1 \leq q := \lfloor \ell \rfloor$, one obviously has

$$\partial_\theta^j f(I,\theta) = \sum_{k\in\mathbb{Z}^n} (i)^{|j|} k_1^{j_1} ... k_n^{j_n} \hat{f}_k(I) e^{ik\cdot\theta} . \tag{G.0.2}$$

From

$$\partial_\theta^j f(I,\theta) := \sum_{k\in\mathbb{Z}^n} (\widehat{\partial^j f})_k(I) e^{ik\cdot\theta} \ , \tag{G.0.3}$$

and by the unicity of Fourier's coefficients one also has

$$\hat{f}_k(I) := \frac{(\widehat{\partial^j f})_k(I)}{k_1^{j_1} ... k_n^{j_n}} . \tag{G.0.4}$$

As in expression (G.0.4) the multi-index $j \in \mathbb{Z}^n$ is arbitrary, for each value of $k \in \mathbb{Z}^n \backslash \{0\}$ we can choose j so that

$$\hat{f}_k(I) = \frac{(\widehat{\partial^j f})_k(I)}{(\max_{i=1,...,n}\{k_i\})^{|j|}} . \tag{G.0.5}$$

Moreover, for any $k \in \mathbb{Z}^n \backslash \{0\}$ one has the trivial inequality

$$\max_{i=1,\dots,n} \{|k_i|\} \geq \frac{|k|}{n} \, .$$

This, together with (G.0.5) and the choice $|j| = q$ yields

$$|\hat{f}_k(I)| = n^{\ell} \frac{|\widehat{(\partial^j f)}_k(I)|}{|k|^q} = n^{\ell} \frac{1/(2\pi)^n |\int_0^{2\pi} \partial^j f(I,\theta) e^{\mathrm{i}k\cdot\theta} d\theta|}{|k|^q} \leq n^{\ell} \frac{|\partial^j f(I,\theta)|}{|k|^q} \, , \tag{G.0.6}$$

which, once the supremum over the actions is taken, implies the result. □

Appendix H

Pöschel's Normal form

Given a function F in $\mathcal{D}_{r,s}$, the notations $\mathcal{P}_\Lambda$ and $\mathcal{P}_K$ stand for the projections

$$\mathcal{P}_\Lambda F(I,\theta) := \sum_{k\in\mathbb{Z}^n : k\in\Lambda} F_k(I)e^{ik\cdot\theta}\ ,\ \ \mathcal{P}_K F(I,\theta) := \sum_{k\in\mathbb{Z}^n : |k|_1\le K} F_k(I)e^{ik\cdot\theta}$$

Accordingly with our notations, we state here the result of Pöschel [104].

Lemma H.0.1 (Poschel's normal form). *Let $\varrho,\sigma>0$ and $\mathbf{H}(I,\theta)=\mathbf{h}(I)+\mathbf{f}(I,\theta)$ be analytic on*

$$\mathcal{D}_{\Lambda,\varrho,\sigma} := \{(I,\theta)\in\mathbb{C}^n : \left|I-D_\Lambda\right|_2<\varrho\ ,\ \ \theta\in\mathbb{T}^n_\sigma\}$$

where D_Λ is (α,K)-nonresonant modulo Λ with respect to the integrable Hamiltonian $\mathbf{h}$. Also, let M denote the hermitian norm of the hessian of $\mathbf{h}$ over $\mathcal{D}_{\Lambda,\varrho,\sigma}$.

If, for some $\varrho'>0$, one is insured

$$||\mathbf{f}||_{\varrho,\sigma}\le\epsilon\le\frac{1}{256\xi}\frac{\alpha\varrho'}{K},\qquad \varrho'\le\left(\varrho,\frac{\alpha}{2\xi MK}\right) \tag{H.0.1}$$

for some $\xi>1$ and

$$K\sigma\ge 6, \tag{H.0.2}$$

then there exists a real-analytic, symplectic transformation $\Psi : \mathcal{D}_{\Lambda,\varrho'/2,\sigma/6}\longrightarrow\mathcal{D}_{\Lambda,\varrho,\sigma}$ taking $\mathbf{H}$ into resonant normal form, that is

$$\mathbf{H}\circ\Psi=\mathbf{h}+\mathbf{g}+\mathbf{f}^*\ ,\ \ \{\mathbf{h},\mathbf{g}\}=0\,. \tag{H.0.3}$$

Moreover, denoting by $\mathbf{g_0} := P_\Lambda P_K\mathbf{f}$ the resonant part of $\mathbf{f}$, we have the estimates

$$||\mathbf{g}-\mathbf{g_0}||_{\varrho'/2,\sigma/6}\le 64\frac{K}{\alpha\varrho'}\epsilon^2\ ,\ \ ||\mathbf{f}^*||_{\varrho'/2,\sigma/6}\le e^{-K\sigma/6}\epsilon. \tag{H.0.4}$$

Furthermore, Ψ is close to the identity, in the sense that, for any $(I,\theta)\in\mathcal{D}_{\Lambda,\varrho'/2,\sigma/6}$, one has

$$\frac{\left|\Pi_I\Psi-I\right|_2}{\rho'}\le 2^3\frac{K}{\alpha\rho'}\epsilon\le\frac{1}{32\xi}\quad ,\qquad \frac{\left|\Pi_\theta\Psi-\theta\right|_\infty}{\sigma}\le\frac{2^5K}{3\alpha\rho'}\epsilon\le\frac{1}{24\xi} \tag{H.0.5}$$

where Π_I,Π_θ denote the projection on the action and angle variables, respectively.

Appendix I

Morse's functions, Sard's Theorem, and a quantitative local inversion theorem

Morse's functions and Sard's Theorem

We refer to [55]- [68] for more details.

Let V be an open subset of the euclidean space $\mathbb{R}^n$, and let f be a function of class C^2 over V.

Definition I.0.1. We say that f has a non-degenerate critical point at $a \in V$ if the Hessian form $\mathsf{H}_f(a)$ is nondegenerate. f is said to be a Morse function on V if all its critical points are non-degenerate.

Morse's functions are generic in the following sense

Proposition I.0.1. *The parameters $\lambda \in \mathbb{R}^n$ such that $f(x) - \lambda \cdot x$ is a Morse function are contained in a dense residual subset of measure zero in $\mathbb{R}^n$.*

For the sake of completeness, we also state

Theorem I.0.1 (Sard)**.** *Let X, Y be differentiable manifolds. The set of critical values of a smooth map $f : X \longrightarrow Y$ has measure zero.*

Quantitative local inversion Theorem

The following statement can be found in ref. [48] (Theorem B.1).

Lemma I.0.1 (Quantitative local inversion theorem)**.**
Let $\mathcal{A}$ be a convex subset of $\mathbb{C}^n$ and $f \in C^1(\mathcal{A}; \mathbb{C}^n)$. Suppose that at a point $x_0 \in \mathcal{A}$ the Jacobian of f is invertible and assume

$$\varrho := \sup_{x \in \mathcal{A}} ||\mathbb{I} - (Df(x_0))^{-1} Df(x)|| < 1 \,.$$

Then, the Jacobian Df is invertible in $\mathcal{A}$ and

$$||(Df(x))^{-1}|| \leq \frac{||(Df(x_0))^{-1}||}{1-\varrho} \,.$$

Moreover, f is injective on $\mathcal{A}$ and the Lipschitz constant L_φ of its inverse function $\varphi : f(\mathcal{A}) \longrightarrow \mathcal{A}$ satisfies

$$L_\varphi \leq \frac{||(Df(x_0))^{-1}||}{1-\varrho}$$

on $f(\mathcal{A})$.

In addition, if $\mathcal{A} := \mathbb{B}_r(x_0)$, $\zeta := \dfrac{r(1-\varrho)}{||Df^{-1}(x_0)||}$ and $y_0 = f(x_0)$, then

$$\mathbb{B}_\zeta(y_0) \subset f(\mathcal{A}) \,.$$